Quantum Field Theory Notes

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Here is also an index: genindex.html.
This chapter is about the preliminary knowledge required by this topic.

1.1 Questions
2.1 From Particles to Fields

In this chapter we discuss the Lagrangian formalism of particles and fields.

1. Lagrangian and Euler-Lagrangian equation of single particle;
2. Generalize single particle formalism to fields;
3. Using the least action principle to derive some famous equation of motions: Klein-Gordon equation, Dirac equation, Maxwell equation, and equation for massive vector particles.
4. Relation to statistical mechanics.

2.1.1 Classical Mechanics

For single particle, we define a Lagrangian $L(q_i, \dot{q}_i, t)$ which is a function of generalized coordinate $q_i$, its derivative, and time $t$. To be precise, the generalized coordinates could be time dependent.

Action is defined as the integral of Lagrangian over time,

$$S = \int dt L(q_i, \dot{q}_i, t).$$

The principle that leads to the equation of motion that the particle obeys is the one that extremize the action. Mathematically,

$$\delta S = 0,$$

which gives us the Euler-Lagrangian equation,

$$\partial_t \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0.$$

Derivation of Euler-Lagrangian Equation

- [ ] Here goes the derivation

On the other hand, a Legendre transform of the Lagrangian is the Hamiltonian. For single particle

$$H = \dot{q}p - L,$$
where the generalized momentum is \( p = \frac{\partial L}{\partial \dot{q}} \). The equation of motion starting from Hamilton is

\[
\begin{align*}
\dot{q} &= \frac{\partial H}{\partial p}, \\
\dot{p} &= -\frac{\partial H}{\partial q}.
\end{align*}
\]

### 2.1.2 Generalization to Fields

For single particle, the dynamics is described by where the particle is at a certain time. However, field spans over space and evolve over time. Thus we define a Lagrangian \( \mathcal{L} \) at each space point, which should be called Lagrangian density. Integrate over space we find the Lagrangian

\[
L = \int d^3x \mathcal{L}.
\]

To be more specific, Lagrangian density is a function of \( \phi_i(x^\mu), \phi_{i+}(x^\mu), \mu \), where \( x^\mu \) are the space time coordinates. Then

\[
L(t) = \int d^3x \mathcal{L}(\phi_i(x^\mu), \phi_{i+}(x^\mu), \mu).
\]

#### Lagrangian and Lagrangian Density

In quantum field theory, Lagrangian density is usually called Lagrangian.

Similar to single particle theory, action is defined as integral of Lagrangian over time,

\[
S = \int dt L(t).
\]

Apply the same principle as the single particle case, we can derive the Euler-Lagrangian for the fields

\[
\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_0 \phi_i)} \right) - \frac{\partial \mathcal{L}}{\partial \phi_i} = 0.
\]

#### Derivation of Euler-Lagrangian Equation

- Here goes the derivation

We can also define a momentum of field \( \phi_i \),

\[
\Pi(x^\mu) = \frac{\partial \mathcal{L}}{\partial (\partial_0 \phi_i)}.
\]

The Hamiltonian density follows

\[
\mathcal{H} = \dot{\phi}_i \Pi - \mathcal{L}.
\]
2.1.3 Examples of Lagrangian

Klein-Gordon Equation

Dirac Equation

Maxwell Equation

Equation for Massive Vector Particles

Shrodinger Equation

2.1.4 Refs and Notes


2.2 Noether’s Theorem

Noether’s theorem deals with continuous symmetry.

2.2.1 Noether’s Theorem of Particles

2.2.2 Noether’s Theorem of Fields

Suppose we have a continuous transformation, which is internal, that transforms the fields according to

\[ \phi_i(x^\mu) \rightarrow \phi_i(x^\mu) + \delta\phi_i(x^\mu). \]

For convenience, we explicitly write the variation \( \delta\phi_i(x^\mu) \) as a continuous quantity \( \alpha \), i.e.,

\[ \delta\phi_i(x^\mu) = \alpha \Delta \phi(x^\mu). \]

Noether’s theorem states that if this continuous preserves the Lagrangian, we can define conserved Noether current thus conserved charge.

Planning the Proof

1. Write down the variation of Lagrangian.
2. Combine the terms and apply the Euler-Lagrangian equation.
3. If the Lagrangian is invariant under such a continuous tranformation, blablaba.

The variation of Lagrangian is

\[ \delta\mathcal{L}(\phi_i, \dot{\phi}_i) = \frac{\partial\mathcal{L}}{\partial\phi_i}\delta\phi_i + \frac{\partial\mathcal{L}}{\partial\phi_{i,\mu}}\delta(\phi_{i,\mu}). \]  

(2.1)

We know that the variation and partial derivative can be exchanged, such that

\[ \delta(\phi_{i,\mu}) = \partial_{\mu}(\delta\phi_i). \]
We rewrite the second term in Eq. (2.1),

\[
\frac{\partial L}{\partial \phi_{i,\mu}} \delta(\phi_{i,\mu}) = \frac{\partial L}{\partial \phi_{i,\mu}} \delta \phi_i
\]

\[= \partial \mu \left( \frac{\partial L}{\partial \phi_{i,\mu}} \delta \phi_i \right) - \delta \phi_i \partial \mu \left( \frac{\partial L}{\partial \phi_{i,\mu}} \right).
\]

Plug it back into Eq. (2.1), we have

\[
\delta L(\phi_i, \dot{\phi}_i) = \partial \mu \left( \frac{\partial L}{\partial \phi_{i,\mu}} \delta \phi_i \right).
\]

The first term is zero by Euler-Lagrangian equation. Thus

\[
\delta L(\phi_i, \dot{\phi}_i) = \partial \mu \left( \frac{\partial L}{\partial \phi_{i,\mu}} \delta \phi_i \right).
\]

Now we impose the condition that the Lagrangian is invariant under such a continuous transformation, so that \(\delta L = 0\).

\[
\partial \mu \left( \frac{\partial L}{\partial \phi_{i,\mu}} \delta \phi_i \right) = 0.
\] (2.2)

Eq. (2.2) defines the constant of motion. Put the definition of \(\delta \phi_i\) back in,

\[
\alpha \partial \mu \left( \frac{\partial L}{\partial \phi_{i,\mu}} \Delta \phi_i \right) = 0.
\] (2.3)

### 2.2.3 Examples of Noether Current

#### Global Phase Transformation

For the Lagrangian

\[
L = \partial^\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi,
\]

that leads to Klein-Gordon equation, a transformation

\[
\phi \rightarrow e^{i \alpha} \phi, \\
\phi^* \rightarrow e^{-i \alpha} \phi^*,
\]

will not change the scalar particle Lagrangian.

The corresponding Noether current is defined by

\[
\partial \mu j^\mu = 0,
\]

where

\[
j^\mu = -i(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*).
\]
Along with the current we find the conserved charge \[ Q = \int d^3x j^0, \]
which satisfies \[ \frac{\partial Q}{\partial t} = 0. \]

**Proof**

Here is the proof.

**Space-time Translation**

For arbitrary Lagrangian \( \mathcal{L}(x^\mu) \) which is space-time dependent, we can calculate the action

\[ S = \int d^4x \mathcal{L}. \]

If the action is invariant under space-time translation

\[ x^\mu \rightarrow x^\mu + \alpha a^\mu, \]

we find the conserved current to be the energy-momentum tensor \( T^{\mu\nu} \)

\[ T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi)} \partial^\nu \phi - g^{\mu\nu} \mathcal{L}. \]

The corresponding conservation equation is

\[ \partial_\mu T^{\mu\nu} = 0, \]

which defines the four charges

\[ Q^\mu = \int d^3T^{\mu\nu}. \]

**Proof Energy-momentum Tensor as Noether Current**

QED.

For the Lagrangian

\[ \mathcal{L} = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2, \]

one can easily prove that the corresponding energy-momentum tensor is

\[ T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - g^{\mu\nu} \mathcal{L}. \]
Derivation of Energy-momentum for Real Scalar Lagrangian

QED.

The 00 component is in fact the Hamiltonian density $\mathcal{H}$.

**Prove that** $T^{00} = \mathcal{H}$

Calculate $T^{00}$.

$$T^{00} = \partial^0 \phi \partial^0 \phi - \mathcal{L}$$

$$= \frac{1}{2} (\partial^0 \phi \partial^0 \phi + \partial^i \phi \partial^i \phi + m^2 \phi^2).$$

Notice that the Hamiltonian density is

$$\mathcal{H} = \Pi \partial^0 \phi - \mathcal{L},$$

where

$$\Pi = \frac{\partial \mathcal{L}}{\partial (\partial^0 \phi)} = \partial^0 \phi.$$

Plug in the momentum we find

$$\mathcal{H} = \partial^0 \phi \partial^0 \phi - \mathcal{L} = T^{00}.$$

**Dilation and Noether Current**

Dilation can be written as

$$x_\mu \rightarrow ax_\mu,$$

$$\phi \rightarrow a^{-1} \phi.$$

The Noether current corresponding to such transformation is

$$j_D^\mu = T^{\mu \rho} x_\rho + \frac{1}{2} \partial^\mu \phi^2.$$

Notice that Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4!} \lambda \phi^4,$$

which is $\phi^4$ theory, is invariant under dilation.

**2.2.4 References and Notes**

**2.3 Gauge Symmetries**

**2.3.1 U(1) Global Gauge Invariance**

Assuming the Lagrangian is invariant under U(1) global gauge transformation

$$\psi(x) \rightarrow e^{i\alpha} \psi(x),$$
we obtain the Noether current

\[ j^\mu = -e \bar{\psi} \gamma^\mu \psi, \]

which is conserved

\[ \partial_\mu j^\mu = 0. \]

### 2.3.2 U(1) Local Gauge Invariance

Introduce the U(1) local gauge transformation

\[ \psi(x) \rightarrow e^{i\alpha(x)} \psi(x) \]

to the Lagrangian, we notice that the Lagrangian is generally not invariant. However, if we require it to be invariant, a new field \( A_\mu \) should be introduced, so that

\[ A_\mu \rightarrow A_\mu + \frac{1}{2} \partial_\mu \alpha(x). \]

The way this new field comes into the Lagrangian is to replace all the derivatives \( _\mu \) with \( D_\mu \),

\[ D_\mu = \partial_\mu - ieA_\mu. \]

What about the kinetic term for this new field? It is constructed from the field strength tensor

\[ F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \]

so that the kinetic term is

\[ -\frac{1}{4} F_{\mu \nu} F^{\mu \nu}. \]

Finally we reach the new (QED) Lagrangian that is invariant under U(1) gauge transformation

\[ \mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + e\bar{\psi} \gamma^\mu \partial_\mu \psi - \frac{1}{4} F_{\mu \nu} F^{\mu \nu}. \]

### 2.3.3 Non-Abelian

Introduce the non-abelian local gauge transformation

\[ q(x) \rightarrow e^{i\alpha_a(x) T_a} q(x). \]

The Lagrangian that is invariant under this transformation is

\[ =\bar{q}(i\gamma^\mu \partial_\mu - m)q - g(\bar{q} \gamma^\mu T_a q)G^a_\mu - \frac{1}{4} G_\mu^a G^{\mu \nu}_a. \]

2.3. Gauge Symmetries
2.4 Symmetry Breaking

Introducing mass to gauge bosons will break the U(1) local gauge symmetry.

**But wait, why preserving U(1) symmetry?**

1. But wait, why do we need to preserve U(1) symmetry?
2. We introduce mass to bosons regardless of the breaking of U(1) symmetry.
3. Such a mass term will lead to infinity in e-e scattering loop. (By counting the orders of moment, we know this is not finite.)
4. Introduce cut-off to momentum.
5. Still have infinity in higher order loop diagrams.
6. Introduce infinite numbers of cut-off for the infinite infinities of infinite higher order loop diagrams.
7. WTF.

So we can not just simply introduce the mass term.

2.4.1 Spontaneous Breaking of Gauge Symmetry

We can not simply write a mass term in the Lagrangian. However, mass is simply an interaction term quadratic in the field, e.g., $-m^2 \phi^2$. Geometrically speaking, we need a upward opening quadratic potential of the field, i.e., Fig. 2.1.

The point is, since we are talking about the perturbation around the minima, we can also construct a potential with more complicated shape but is has quadratic term around the minima, e.g., Fig. 2.2.

**Spontaneous Breaking Global Gauge Symmetry**

Mathematically the simplest Lagrangian for a real scalar field that could possibly have a mass that we can think of is

$$
\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \mu^2 \phi^2 - \frac{1}{4} \lambda \phi^4,
$$

where we set $\mu^2 < 0$ and $\lambda > 0$.

Notice that we have global gauge symmetry holds. By intuition we expect the field to have a quadratic shape around the minima since this is a quartic potential. Indeed we Taylor expand the potential around the right minimum $\phi_0$, and define a new field around the minimum $\eta(x) = \phi(x) - \phi_0$. The Lagrangian becomes

$$
\mathcal{L}' = \frac{1}{2} \partial_\mu \eta \partial^\mu \eta - \lambda \phi_0^4 \eta^2 + \text{higher order terms} + \text{constant},
$$

where we spot the quadratic term $-\lambda \phi_0^4 \eta^2$ so that mass of the field $\eta$ around its vacuum (minimum) can be defined as $m^2 \equiv \lambda \phi_0^2$.

The two Lagrangians describes the samething around the minimum point $\phi_0$. We notice that the global gauge invariance is broken for field $\eta$.

**What is broken?**

Both Lagrangian $\mathcal{L}$ and $\mathcal{L}'$ are invariant under the global gauge transformation of field $\phi$. But they are all changed under a gauge transformation of field $\eta$. 
Fig. 2.1: To have mass we need a potential with parabola shape at least around the minimum which is the vacuum. Another requirement is that the potential should be bounded.
Fig. 2.2: This potential also have a parabola shape around the minima.
What does that mean?

Since our particle physics are always dealing with small amount of particles, the physics is described by a field near the minimum potential point, which can be called vacuum of the field. When we are doing experiments near the vacuum, we do not see the big picture that the whole potential is quartic.

**That is to say, we are talking about a field \( \eta \) that varying around a minimum potential point.** What we see is a small parabola (quadratic terms) with some corrections (higher order terms). Any potential that can give us a parabola like potential at its minimum will possibly work.

The same trick can be done for complex fields. The Lagrangian we are interested in is

\[
\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - \mu^2 \phi^* \phi - \lambda (\phi^* \phi)^2,
\]

where we also have \( \mu^2 < 0 \) and \( \lambda > 0 \). The potential is a Mexican hat, c.f. Fig. 2.3.

Fig. 2.3: Density plot of the quartic potential. Darker color means smaller potential while brighter color means larger potential. The ring is where the potential minimum is located.

**Mathematica code for the density plot**

```mathematica
range = 1.5; DensityPlot[Log[1000 (1/4 (x^2 + y^2)^2 - 1/2 (x^2 + y^2) + 1/4)], {x, -range, range}, {y, -range, range}, Frame -> False, PlotRange -> Full, ColorFunction -> GrayLevel("SunsetColors"), PerformanceGoal -> "Quality", ColorFunctionScaling -> True, PlotPoints -> 300, Axes -> True, AxesLabel -> {"Re[\Phi]", "Im[\Phi]"}, Ticks -> None, ImageSize -> Large]
```

Since we have two degrees of freedom, we can look at two directions. The direction that circles the ring is the direction that the potential has no change, while the perturbation in the radial direction needs to climb on and off potentials. The potential in the radial direction has a quadratic term around minima. The radial direction perturbation generates mass, however the circular direction perturbation doesn’t generate mass. The massless boson is Goldstone boson, **which becomes a problem because we get a massless mode of the particle.**

Similar picture can be established for three component fields, which has a potential minimum on a 3D spherical surface. The perturbation around that minimum needs three degrees of freedom, thus generating 2 massless Goldstone bosons and one massive boson and one massive boson in the perturbation of radial direction.

### 2.4.2 Spontaneous Breaking of Local Gauge Symmetry

To keep the local gauge invariance, we write the Lagrangian as

\[
D_\mu \phi^* D^\mu \phi - \frac{1}{4} F_{\mu \nu} F^{\mu \nu},
\]

where covariant derivative is \( D_\mu = \partial_\mu - ieA_\mu \). Perform the trick of quartic potential, the Lagrangian becomes

\[
\mathcal{L} = D_\mu \phi^* D^\mu \phi - \mu^2 \phi^* \phi - \lambda (\phi^* \phi)^2 - \frac{1}{4} F_{\mu \nu} F^{\mu \nu}.
\]

**Why a negative sign for the kinetic term of gauge field \(- \frac{1}{4} F_{\mu \nu} F^{\mu \nu}\)?**

Quick and simple answer:

Check the components that contains the time derivative of the field \( A_\mu \).
Local gauge transformation

\[ \phi \rightarrow e^{i\alpha(x)}\phi, \]
\[ A_\mu \rightarrow A_\mu + \frac{1}{e} \partial_\mu \alpha. \]

To work out the perturbation theory, we take the old school treatment,
\[ \phi = \frac{1}{\sqrt{2}}(\phi_0 + \eta + i\xi). \]

The expanded Lagrangian is
\[ \mathcal{L}' = \frac{1}{2} \partial_\mu \eta \partial^\mu \eta + \frac{1}{2} \partial_\mu \xi \partial^\mu \xi - \phi_0^2 A_\mu A^\mu - e\phi_0 A_\mu \partial^\mu \xi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \text{other terms}. \]

We are super happy to locate the terms \(-\phi_0^2 \lambda \eta^2\) and \(\frac{1}{2} e^2 \phi_0^2 A_\mu A^\mu\) since they give mass to \(\eta\) field and \(A_\mu\) gauge field.

**Problems**

However, we still have a massless Goldstone boson \(m\xi = 0\).

Meanwhile, we notice that the term \(-e\phi_0 A_\mu \partial^\mu \xi\) is weird because we have a **direct transformation from one field to another**!

**HINT**

Or maybe not? We do not expect such direct tranformation. But maybe they are actually the SAME field! By interpreting the \(\xi\) field as part of the gauge field can eliminate the massless Goldstone boson.

To support such conjecture, we need to choose a different set of real field instead of the original \(\eta, \xi, \) and \(A_\mu\). We have been using Cartesian coordinates since the very beginning of this trick. But polar coordinates are better for spherically symmetric potential. So we rewrite the field as
\[ \phi = \frac{1}{\sqrt{2}}(\phi_0 + h(x)) e^{i\theta(x)/\phi_0}, \]
\[ A'_\mu = A_\mu + \frac{1}{e\phi_0} \partial_\mu \theta, \]

where \(h, \theta\) are real and \(\phi_0\) is where the minimum potential is reached.

A miracle happens immediately,
\[ \mathcal{L}' = \frac{1}{2} \partial_\mu h \partial^\mu h - \lambda h^2 + \frac{1}{2} e^2 \phi_0^2 A'_\mu A'^\mu + \frac{1}{2} e^2 A'_\mu A'^\mu - \lambda h^2 - \frac{1}{4} \lambda h^4 + \phi_0 e^2 A'_\mu A'^\mu h - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \]

The field \(\theta\) is gone. Meanwhile the two real fields \(h\) and \(A'_\mu\) gain mass.

**What is the magic?**

The magic is that the massless Goldstone boson by field \(\xi\) becomes part of the gauge field \(A_\mu\).

Mathematically, we notice that the two expansion are related to each other under the condition of perturbation in both fields
\[ \frac{1}{\sqrt{2}}(\phi_0 + h) e^{i\theta/\phi_0} = \frac{1}{\sqrt{2}}(\phi_0 + h)(1 + i\frac{\theta}{\phi_0}) = \frac{1}{\sqrt{2}}(\phi_0 + h + i\theta + i\frac{h\theta}{\phi_0}). \]
where the last term \( i \frac{h}{\partial \phi} \) is higher order in the perturbation fields so we drop it. Field \( h \) corresponds to \( \eta \) while field \( \theta \) corresponds to \( \xi \).

However, the difference is that here we do not need to perturb the field \( \theta \) since it’s gone in the Lagrangian. And it should NOT be in there in principle. The perturbation method using \( \eta \) and \( \xi \) introduced a spurious field.

### 2.4.3 References and Notes

1. Halzen & Martin
In classical mechanics, we have Poisson brackets, which is replaced by commutation relation in quantum mechanics. By promoting quantities into operator and introduce commutation relations, we have quantization. Similar techniques can be applied to fields.

3.1 Classical Mechanics and Quantum Mechanics

Throughout this section, we use Einsten’s summation rule.

3.1.1 Poisson Brackets in Classical Mechanics

Definition of Poisson Bracket

Poisson bracket is defined as

\[
\{ A, B \} = \frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i}.
\]

In classical mechanics, we can find conserved quantities without the help of symmetries, by using Hamilton’s form. For an conserved quantity \( A(q_i, p_i) \), we have

\[
\partial_t A(q_i, p_i) = 0.
\]

By using Hamilton’s equations, we can find that

\[
\partial_t A(q_i, p_i) = \{ A(q_i, p_i), H \}.
\]

Thus the conservation condition is

\[
\{ A(q_i, p_i), H \} = 0.
\]

Derivation of the Relation between Conserved Quantities and Poisson Bracket

On the other hand,

\[
\partial_t A(q_i, p_i) = \frac{\partial A(q_i, p_i)}{\partial q_i} \partial_t q_i + \frac{\partial A(q_i, p_i)}{\partial p_i} \partial_t p_i. \tag{3.1}
\]
Recall that the Hamilton’s equations are

\[ \partial_t q_i = \frac{\partial H}{\partial p_i}, \]
\[ \partial_t p_i = -\frac{\partial H}{\partial q_i}. \]

Plug them back into Eq. (3.1), we have

\[ \partial_t A(q_i, p_i) = \frac{\partial A(q_i, p_i)}{\partial q_i} \frac{\partial H}{\partial p_i} + \frac{\partial A(q_i, p_i)}{\partial p_i} \left( -\frac{\partial H}{\partial q_i} \right) = \{A(q_i, p_i), H\}. \]

**Self-consistency Check of Poisson Bracket and Conserved Quantities**

We can easily find the Poisson brackets \( \{q_i, H\} \) and \( \{p_i, H\} \), which are in fact \( \partial q_i \) and \( \partial p_i \).

We would actually get the Hamilton’s equations.

### 3.2 Klein-Gordon Field
Weak Interaction

4.1 Weinberg-Salam Model

Electroweak interaction is determined by local gauge invariance of Lagrangian. Specifically, we have two types of interactions

\[-igJ_\mu \cdot W^\mu = -ig\bar{\xi}_L \gamma_\mu T \cdot W^\mu \xi_L\]
\[-ig'j^Y_\mu B^\mu = -ig'\bar{\psi}\gamma_\mu \frac{Y}{2}\psi B^\mu,\]

where \(T\) and \(Y\) are generators of \(SU(2)_L\) and \(U(1)\). The doublet \(\xi_L\) and singlet \(\psi\) fermions and bosons.

For a group multiplication \(G = SU(2)_L \times U(1)\), the generators are related

\[Q = T^3 + \frac{Y}{2},\]

where \(Q\) is the generator of group \(G\).

Then we can write down the currents

\[j_{\text{em}}^\mu = j^3_\mu + \frac{1}{2}j^Y_\mu.\]

Since we know the neutral currents \(j^3_\mu\) and \(\frac{1}{2}j^Y_\mu\), we can calculate the EM current by adding them up. Hence we actually can relate \(g\) and \(g'\) by looking at the coefficient of \(A^\mu\) field. More specific proof of this is to use the Higgs field and find the actual electromagnetic field \(A\)

**Why does the Weinberg-Salam model work**

The choice of vacuum in Weinberg-Salam model is quite unique.
5.1 Supersymmetric Quantum Mechanics

For a quantum mechanical system, it is always possible to find a partner potential. Suppose we have a Hamiltonian $H_1$ in quantum mechanics, which is defined as

$$H_1 = -\frac{\hbar}{2m} \partial_x^2 + V_1(x).$$

By decomposing it into

$$H_1 = A^\dagger A,$$

we can define the two operators

$$A = \frac{\hbar}{\sqrt{2m}} \partial_x + W(x),$$

$$A^\dagger = -\frac{\hbar}{\sqrt{2m}} \partial_x + W(x).$$

The quantity $W(x)$ is called superpotential. The partner Hamiltonian of $H_1$ is

$$H_2 = AA^\dagger,$$

using which one finds the corresponding potential is

$$V_2(x) = W(x)^2 - \frac{\hbar}{\sqrt{2m}} \partial_x W(x).$$

The interesting part is that the superpotential is closely related to ground state wave functions of the original system

$$W(x) = -\frac{\hbar}{\sqrt{2m}} \frac{\psi''_0(x)}{\psi_0(x)}.$$
One can also find the relation between the new wave function $\psi^{(2)}$ and the original one $\psi^{(1)}$,

$$
\psi^{(2)}_n = \frac{1}{\sqrt{E^{(1)}_{n+1}}} A \psi^{(1)}_{n+1},
$$

$$
\psi^{(1)}_{n+1} = \frac{1}{\sqrt{E^{(2)}_n}} A^\dagger \psi^{(2)}_n.
$$

Meanwhile the energy levels are also related

$$
E^{(1)}_{n+1} = E^{(2)}_n.
$$

Proof

QED.

5.1.1 References and Notes

1. An Introduction to Supersymmetry in Quantum Mechanical Systems by T. Wellman
CHAPTER 6

References
Acknowledgement