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What is KryPy and where is the code?

KryPy is a Krylov subspace methods package for Python. If you’re looking for the source code or bug reports, take a look at KryPy’s github page. These pages provide the documentation of KryPy’s API.
2.1 krypy Package

2.1.1 krypy.linsys - Linear Algebraic Systems Solver

The linsys module provides functions for the solution of linear algebraic systems.

```python
class krypy.linsys.LinearSystem(A, b, M=None, Minv=None, Ml=None, Mr=None, ip_B=None, normal=None, self_adjoint=False, positive_definite=False, exact_solution=None)
```

Bases: object

Representation of a (preconditioned) linear system.

Represents a linear system

\[ Ax = b \]

or a preconditioned linear system

\[ M M_l A M_r y = M M_l b \quad \text{with} \quad x = M_r y. \]

Parameters

- `A` – a linear operator on \( \mathbb{C}^N \) (has to be compatible with `get_linearoperator()`).
- `b` – the right hand side in \( \mathbb{C}^N \), i.e., \( b.\text{shape} == (N, 1) \).
- `M` – (optional) a self-adjoint and positive definite preconditioner, linear operator on \( \mathbb{C}^N \) with respect to the inner product defined by `ip_B`. This preconditioner changes the inner product to \( \langle x, y \rangle_M = \langle M x, y \rangle \) where \( \langle \cdot, \cdot \rangle \) is the inner product defined by the parameter `ip_B`. Defaults to the identity.
- `Minv` – (optional) the inverse of the preconditioner provided by `M`. This operator is needed, e.g., for orthonormalizing vectors for the computation of Ritz vectors in deflated methods.
- `Ml` – (optional) left preconditioner, linear operator on \( \mathbb{C}^N \). Defaults to the identity.
- `Mr` – (optional) right preconditioner, linear operator on \( \mathbb{C}^N \). Defaults to the identity.
- `ip_B` – (optional) defines the inner product, see `inner()`.
- `normal` – (bool, optional) Is \( M_l A M_r \) normal in the inner product defined by `ip_B`? Defaults to `False`.
- `self_adjoint` – (bool, optional) Is \( M_l A M_r \) self-adjoint in the inner product defined by `ip_B`? `self_adjoint=True` also sets `normal=True`. Defaults to `False`.
- `positive_definite` – (bool, optional) Is \( M_l A M_r \) positive (semi-)definite with respect to the inner product defined by `ip_B`? Defaults to `False`.
• **exact_solution** – (optional) If an exact solution $x$ is known, it can be provided as a `numpy.array` with `exact_solution.shape == (N, 1)`. Then error norms can be computed (for debugging or research purposes). Defaults to None.

```python
MMlb_norm = None
Norm of the right hand side.

\|MMlb(b)\|_M^{-1}
```

```python
N = None
Dimension $N$ of the space $\mathbb{C}^N$ where the linear system is defined.
```

```python
get_ip_Minv_B()
Returns the inner product that is implicitly used with the positive definite preconditioner $M$.
```

```python
get_residual(z, compute_norm=False)
Compute residual.
For a given $z \in \mathbb{C}^N$, the residual

$$r = MMlb(b - Az)$$

is computed. If `compute_norm == True`, then also the absolute residual norm

$$\|MMlb(b - Az)\|_M^{-1}$$

is computed.
```

**Parameters**

* `z` – approximate solution with `z.shape == (N, 1)`.
* `compute_norm` – (bool, optional) pass `True` if also the norm of the residual should be computed.

class krypy.linsys.Cg(linear_system, **kwargs)
Bases: krypy.linsys._KrylovSolver
```

Preconditioned CG method.

The **preconditioned conjugate gradient method** can be used to solve a system of linear algebraic equations where the linear operator is self-adjoint and positive definite. Let the following linear algebraic system be given:

$$MMlb AMr y = MMlb b,$$

where $x = Mry$ and $MMlb AMr$ is self-adjoint and positive definite with respect to the inner product $\langle \cdot, \cdot \rangle$ defined by `ip_B`. The preconditioned CG method then computes (in exact arithmetics!) iterates $x_k \in x_0 + Mr K_k$ with $K_k := K_k(MMlb AMr, r_0)$ such that

$$\|x - x_k\|_A = \min_{z \in x_0 + Mr K_k} \|x - z\|_A.$$

The Lanczos algorithm is used with the operator $MMlb AMr$ and the inner product defined by $\langle x, y \rangle_M^{-1} = \langle M^{-1} x, y \rangle$. The initial vector for Lanczos is $r_0 = MMlb(b - A x_0)$ - note that $Mr$ is not used for the initial vector.

Memory consumption is:

* `if store_arnoldi==False: 3 vectors or 6 vectors if $M$ is used.

* `if store_arnoldi==True: about maxiter+1 vectors for the Lanczos basis. If $M$ is used the memory consumption is 2*(maxiter+1).

**Caution:** CG’s convergence may be delayed significantly due to round-off errors, cf. chapter 5.9 in [LieS13].

All parameters of `_KrylovSolver` are valid in this solver. Note the restrictions on $M, Mlb, A, Mr$ and `ip_B` above.
static operations (nsteps)

Returns the number of operations needed for nsteps of CG

class krypy.linsys.Minres(linear_system, ortho='lanczos', **kwargs)

Bases: krypy.linsys._KrylovSolver

Preconditioned MINRES method.

The preconditioned minimal residual method can be used to solve a system of linear algebraic equations where the linear operator is self-adjoint. Let the following linear algebraic system be given:

\[ M_i A M_r y = M_i b, \]

where \( x = M_r y \) and \( M_i A M_r \) is self-adjoint with respect to the inner product \( \langle \cdot, \cdot \rangle \) defined by \texttt{inner_product}. The preconditioned MINRES method then computes (in exact arithmetics!) iterates \( x_k \in x_0 + M_r K_k \) with \( K_k := K_k(M_i A M_r, r_0) \) such that

\[ \| M_i(b - A x_k)\|_{M^{-1}} = \min_{z \in x_0 + M_r K_k} \| M_i(b - A z)\|_{M^{-1}}. \]

The Lanczos algorithm is used with the operator \( M_i A M_r \) and the inner product defined by \( \langle x, y \rangle_{M^{-1}} = \langle M^{-1}x, y \rangle \). The initial vector for Lanczos is \( r_0 = M_i(b - A x_0) \) - note that \( M_r \) is not used for the initial vector.

Memory consumption is:

- if \texttt{store_arnoldi=False}: 3 vectors or 6 vectors if \( M \) is used.
- if \texttt{store_arnoldi=True}: about maxiter+1 vectors for the Lanczos basis. If \( M \) is used the memory consumption is \( 2 \times (\text{maxiter}+1) \).

Caution: MINRES’ convergence may be delayed significantly or even stagnate due to round-off errors, cf. chapter 5.9 in [LieS13].

In addition to the attributes described in \_KrylovSolver, the following attributes are available in an instance of this solver:

- \texttt{lanczos}: the Lanczos relation (an instance of \texttt{Arnoldi}).

All parameters of \_KrylovSolver are valid in this solver. Note the restrictions on \( M, M_i, A, M_r \) and \texttt{ip_B} above.

static operations (nsteps)

Returns the number of operations needed for nsteps of MINRES

class krypy.linsys.Gmres(linear_system, ortho='mgs', **kwargs)

Bases: krypy.linsys._KrylovSolver

Preconditioned GMRES method.

The preconditioned generalized minimal residual method can be used to solve a system of linear algebraic equations. Let the following linear algebraic system be given:

\[ M_i A M_r y = M_i b, \]

where \( x = M_r y \). The preconditioned GMRES method then computes (in exact arithmetics!) iterates \( x_k \in x_0 + M_r K_k \) with \( K_k := K_k(M_i A M_r, r_0) \) such that

\[ \| M_i(b - A x_k)\|_{M^{-1}} = \min_{z \in x_0 + M_r K_k} \| M_i(b - A z)\|_{M^{-1}}. \]

The Arnoldi algorithm is used with the operator \( M_i A M_r \) and the inner product defined by \( \langle x, y \rangle_{M^{-1}} = \langle M^{-1}x, y \rangle \). The initial vector for Arnoldi is \( r_0 = M_i(b - A x_0) \) - note that \( M_r \) is not used for the initial vector.

Memory consumption is about maxiter+1 vectors for the Arnoldi basis. If \( M \) is used the memory consumption is \( 2 \times (\text{maxiter}+1) \).

If the operator \( M_i A M_r \) is self-adjoint then consider using the MINRES method \texttt{Minres}.

All parameters of \_KrylovSolver are valid in this solver.
static operations \( (n\text{steps}) \)
Returns the number of operations needed for \( n\text{steps} \) of GMRES

class krypy.linsys._KrylovSolver \( (\text{linear\_system}, \ x0=\text{None}, \ \text{tol}=1e-05, \ \text{maxiter}=\text{None}, \ \text{explicit\_residual}=\text{False}, \ \text{store\_arnoldi}=\text{False}, \ \text{dtype}=\text{None}) \)
Prototype of a Krylov subspace method for linear systems.
Init standard attributes and perform checks.
All Krylov subspace solvers in this module are applied to a \textit{LinearSystem}. The specific methods may impose further restrictions on the operators

Parameters

- \texttt{linear\_system} – a \texttt{LinearSystem}.
- \texttt{x0} – (optional) the initial guess to use. Defaults to zero vector. Unless you have a good reason to use a nonzero initial guess you should use the zero vector, cf. chapter 5.8.3 in \textit{Liesen, Strakos. Krylov subspace methods. 2013}. See also \texttt{hegesdus}().
- \texttt{tol} – (optional) the tolerance for the stopping criterion with respect to the relative residual norm:
  \[
  \frac{\|MM_l(b - A(x_0 + Mr_yk))\|_{M^{-1}}}{\|MM_l b\|_{M^{-1}}} \leq \texttt{tol}
  \]
- \texttt{maxiter} – (optional) maximum number of iterations. Defaults to \( N \).
- \texttt{explicit\_residual} – (optional) if set to \texttt{False} (default), the updated residual norm from the used method is used in each iteration. If set to \texttt{True}, the residual is computed explicitly in each iteration and thus requires an additional application of \( M \), \( Ml \), \( A \) and \( Mr \) in each iteration.
- \texttt{store\_arnoldi} – (optional) if set to \texttt{True} then the computed Arnoldi basis and the Hessenberg matrix are set as attributes \( V \) and \( H \) on the returned object. If \( M \) is not \texttt{None}, then also \( P \) is set where \( V=M*P \). \texttt{store\_arnoldi} defaults to \texttt{False}. If the method is based on the Lanczos method (e.g., \texttt{Cg} or \texttt{Minres}), then \( H \) is real, symmetric and tridiagonal.
- \texttt{dtype} – (optional) an optional dtype that is used to determine the dtype for the Arnoldi/Lanczos basis and matrix.

Upon convergence, the instance contains the following attributes:

- \texttt{xk}: the approximate solution \( x_k \).
- \texttt{resnorms}: relative residual norms of all iterations, see parameter \texttt{tol}.
- \texttt{errnorms}: the error norms of all iterations if \texttt{exact\_solution} was provided.
- \( V \), \( H \) and \( P \) if \texttt{store\_arnoldi}==\texttt{True}, see \texttt{store\_arnoldi}

If the solver does not converge, a \texttt{ConvergenceError} is thrown which can be used to examine the misconvergence.

\texttt{errnorms} = \texttt{None}
Error norms.

\texttt{iter} = \texttt{None}
Iteration number.

static operations \( (n\text{steps}) \)
Returns the number of operations needed for \( n\text{steps} \) of the solver.

Parameters \texttt{nsteps} – number of steps.

Returns a dictionary with the same keys as the timings parameter. Each value is the number of operations of the corresponding type for \( n\text{steps} \) iterations of the method.

\texttt{resnorms} = \texttt{None}
Relative residual norms as described for parameter \texttt{tol}.
2.1.2 krypy.deflation - Linear Systems Solvers with Deflation

The deflation module provides functions for deflated Krylov subspace methods. With deflation, a linear system is multiplied with a suitable projection with the goal of accelerating the overall solution process for the linear system. This module provides tools that are needed in deflated Krylov subspace methods such as involved projections and computations of Ritz or harmonic Ritz pairs from a deflated Krylov subspace.

class krypy.deflation.DeflatedCg(*args, **kwargs)
    Bases: krypy.deflation._DeflationMixin, krypy.linsys.Cg

Deflated preconditioned CG method.

See _DeflationMixin and Cg for the documentation of the available parameters.

_class apply_projection (Av)
    Computes \( (C, M_l A M_r V_n) \) efficiently with a three-term recurrence.

class krypy.deflation.DeflatedMinres(linear_system, U=None, projection_kwargs=None, *args, **kwargs)
    Bases: krypy.deflation._DeflationMixin, krypy.linsys.Minres

Deflated preconditioned MINRES method.

See _DeflationMixin and Minres for the documentation of the available parameters.

class krypy.deflation.DeflatedGmres(linear_system, U=None, projection_kwargs=None, *args, **kwargs)
    Bases: krypy.deflation._DeflationMixin, krypy.linsys.Gmres

Deflated preconditioned GMRES method.

See _DeflationMixin and Gmres for the documentation of the available parameters.

class krypy.deflation._DeflationMixin(linear_system, U=None, projection_kwargs=None, *args, **kwargs)
    Bases: object

Mixin class for deflation in Krylov subspace methods.

Can be used to add deflation functionality to any solver from linsys.

Parameters

- **linear_system** – the LinearSystem that should be solved.
- **U** – a basis of the deflation space with \( U.shape == (N, k) \).

All other parameters are passed through to the underlying solver from linsys.

\[
B = \langle V_{n+1}, M_l A M_r U \rangle.
\]

This property is obtained from \( C \) if the operator is self-adjoint. Otherwise, the inner products have to be formed explicitly.

\[
C = None
\]

\[
C = \langle U, M_l A M_r V_n \rangle.
\]

This attribute is updated while the Arnoldi/Lanczos method proceeds. See also _apply_projection() .

\[
E = None
\]

\[
E = \langle U, M_l A M_r U \rangle.
\]

_class apply_projection (Av)
    Apply the projection and store inner product.
Parameters $v$ – the vector resulting from an application of $M_lAM_r$ to the current Arnoldi vector. (CG needs special treatment, here).

```python
_get_initial_residual(x0)
```

Return the projected initial residual.

```python
_get_xk(yk)
```

```python
_solve()
```

```python
estimate_time(nsteps, ndfl, deflweight=1.0)
```

Estimate time needed to run $n$ steps iterations with deflation

Uses timings from `linear_system` if it is an instance of `TimedLinearSystem`. Otherwise, an `OtherError` is raised.

Parameters

- $n$steps – number of iterations.
- $ndfl$ – number of deflation vectors.
- $deflweight$ – (optional) the time for the setup and application of the projection for deflation is multiplied by this factor. This can be used as a counter measure for the evaluation of Ritz vectors. Defaults to 1.

```python
projection = None
```

Projection that is used for deflation.

```python
class krypy.deflation.ObliqueProjection(linear_system, U, qr_reorthos=0, **kwargs)
```

Oblique projection for left deflation.

- $AU$ = None
  
  Result of application of operator to deflation space, i.e., $M_lAM_rU$.

- $MAU$ = None
  
  Result of preconditioned operator to deflation space, i.e., $MM_lAM_rU$.

- $U$ = None
  
  An orthonormalized basis of the deflation space $U$ with respect to provided inner product.

```python
correct(z)
```

Correct the given approximate solution $z$ with respect to the linear system $linear_system$ and the deflation space defined by $U$.

```python
class krypy.deflation._Projection(linear_system, U, **kwargs)
```

Abstract base class of a projection for deflation.

Parameters

- $A$ – the `LinearSystem`.
- $U$ – basis of the deflation space with $U.shape == (N, d)$.

All parameters of `Projection` are valid except $X$ and $Y$.

```python
class krypy.deflation.Ritz(deflated_solver, mode='ritz')
```

Compute Ritz pairs from a deflated Krylov subspace method.

Parameters

- $deflated_solver$ – an instance of a deflated solver.
- $mode$ – (optional)
- ritz (default): compute Ritz pairs.
- harmonic: compute harmonic Ritz pairs.

coeffs = None
Coefficients for Ritz vectors in the basis \([V_n, U]\).

get Explicit_residual \((indices=None)\)
Explicitly computes the Ritz residual.

get Explicit_resnorms \((indices=None)\)
Explicitly computes the Ritz residual norms.

get_vectors \((indices=None)\)
Compute Ritz vectors.

resnorms = None
Residual norms of Ritz pairs.

values = None
Ritz values.

class krypy.deflation.Arnoldifyer (deflated_solver)

Bases: object

Obtain Arnoldi relations for approximate deflated Krylov subspaces.

Parameters deflated_solver – an instance of a deflated solver.

get \((Wt, full=False)\)
Get Arnoldi relation for a deflation subspace choice.

Parameters

  • Wt – the coefficients \(\tilde{W}\) of the deflation vectors in the basis \([V_n, U]\) with \(\tilde{W}.shape \)  
    \(= (n+d, k)\), i.e., the deflation vectors are \(W = [V_n, U]\tilde{W}\). Must fulfill \(\tilde{W}^*\tilde{W} = I_k\).
  • full – (optional) should the full Arnoldi basis and the full perturbation be returned?
    Defaults to False.

Returns

  • Hh: the Hessenberg matrix with \(Hh.shape \)  
    \(= (n+d-k, n+d-k)\).
  • Rh: the perturbation core matrix with \(Rh.shape \)  
    \(= (l, n+d-k)\).
  • q_norm: norm \(\|q\|_2\).
  • vdiff_norm: the norm of the difference of the initial vectors \(\tilde{v} - \hat{v}\).
  • PNAW_norm: norm of the projection \(P_{W^+AW}\).
  • Vh: (if full == True) the Arnoldi basis with \(Vh.shape \)  
    \(= (N, n+d-k)\).
  • F: (if full == True) the perturbation matrix \(F = \tilde{Z}\tilde{R}^*\tilde{V}^* - \hat{V}^*\hat{R}^*\tilde{Z}^*\).

krypy.deflation.bound_pseudo (arnoldifyer, Wt, g_norm=0.0, G_norm=0.0, GW_norm=0.0,  
WGW_norm=0.0, tol=1e-06, pseudo_type='auto',  
pseudo_kwargs=None, delta_n=20, terminate_factor=1.0)

Bound residual norms of next deflated system.

Parameters

  • arnoldifyer – an instance of Arnoldifyer.
  • Wt – coefficients \(\tilde{W} \in \mathbb{C}^{n+d,k}\) of the considered deflation vectors \(W\) for the basis \([V, U]\) where \(V=last_solver.V\) and \(U=last_P.U\), i.e., \(W = [V, U]\tilde{W}\) and \(\tilde{W} = \text{colspan}(\tilde{W})\). Must fulfill \(\tilde{W}^*\tilde{W} = I_k\).
  • g_norm – norm \(\|g\|\) of difference \(g = c - b\) of right hand sides. Has to fulfill \(\|g\| <  \|b\|\).
• **G\_norm** – norm \(\|G\|\) of difference \(G = B - A\) of operators.

• **GW\_norm** – Norm \(\|G|_W\|\) of difference \(G = B - A\) of operators restricted to \(W\).

• **WGW\_norm** – Norm \(\|(W, GW)\|_2\).

• **pseudo\_type** – One of
  - `auto`': determines if \(\hat{H}\) is non-normal, normal or Hermitian and uses the corresponding mode (see other options below).
  - `nonnormal': the pseudospectrum of the Hessenberg matrix \(\hat{H}\) is used (involves one computation of a pseudospectrum)
  - `normal': the pseudospectrum of \(\hat{H}\) is computed efficiently by the union of circles around the eigenvalues.
  - `hermitian': the pseudospectrum of \(\hat{H}\) is computed efficiently by the union of intervals around the eigenvalues.
  - `contain': the pseudospectrum of the extended Hessenberg matrix \(\begin{bmatrix} \hat{H} \\ S_1 \end{bmatrix}\) is used (pseudospectrum has to be re computed for each iteration).
  - `omit': do not compute the pseudospectrum at all and just use the residual bounds from the approximate Krylov subspace.

• **pseudo\_kwargs** – (optional) arguments that are passed to the method that computes the pseudospectrum.

• **terminate\_factor** – (optional) terminate the computation if the ratio of two subsequent residual norms is larger than the provided factor. Defaults to 1.

### 2.1.3 krypy.recycling - Recycling Linear Systems Solvers

The recycling module provides functions for the solution of sequences of linear algebraic systems. Once a linear system has been solved, the generated data is examined and a deflation space is determined automatically for the solution of the next linear system. Several selection strategies are available.

**krypy.recycling.factories - deflation vector factories**

**class** krypy.recycling.factories.RitzFactory

```python
subset_evaluator, subsets_generator=None, mode='ritz', print_results=None)
```

Bases: krypy.recycling.factories._DeflationVectorFactory

Factory of Ritz vectors for automatic recycling.

**Parameters**

• **subset\_evaluator** – an instance of _RitzSubsetEvaluator that evaluates a proposed subset of Ritz vectors for deflation.

• **subsets\_generator** – (optional) an instance of _RitzSubsetsGenerator that generates lists of subsets of Ritz vectors for deflation.

• **print\_results** – (optional) may be one of the following:
  - None: nothing is printed.
  - 'number': the number of selected deflation vectors is printed.
  - 'values': the Ritz values corresponding to the selected Ritz vectors are printed.
  - 'timings': the timings of all evaluated subsets of Ritz vectors are printed.

**get**(deflated\_solver)
class krypy.recycling.factories.RitzFactorySimple(mode='ritz', n_vectors=0, which='sm')

Bases: krypy.recycling.factories._DeflationVectorFactory

Selects a fixed number of Ritz or harmonic Ritz vectors with respect to a prescribed criterion.

**Parameters**

- **mode** – See mode parameter of Ritz.
- **n_vectors** – number of vectors that are chosen. Actual number of deflation vectors may be lower if the number of Ritz pairs is less than n_vectors.
- **which** – the n_vectors Ritz vectors are chosen such that the corresponding Ritz values are the ones with
  - lm: largest magnitude.
  - sm: smallest magnitude.
  - lr: largest real part.
  - sr: smallest real part.
  - li: largest imaginary part.
  - si: smallest imaginary part.
  - smallest_res: smallest Ritz residual norms.

get (solver)

class krypy.recycling.factories.UnionFactory(factories)

Bases: krypy.recycling.factories._DeflationVectorFactory

Combine a list of factories.

**Parameters**

- **factories** – a list of factories derived from _DeflationVectorFactory.

get (solver)

class krypy.recycling.factories._DeflationVectorFactory

Abstract base class for selectors.

get (solver)

Returns numpy.array of shape (N, k)

krypy.recycling.generators - generators for deflation vector candidates

class krypy.recycling.generators.RitzExtremal(max_vectors=inf)

Bases: krypy.recycling.generators._RitzSubsetsGenerator

Successively returns the extremal Ritz values.

For self-adjoint problems, the indices of the minimal negative, maximal negative, minimal positive and maximal positive Ritz values are returned.

For non-self-adjoint problems, only the indices of the Ritz values of smallest and largest magnitude are returned.

generate (ritz, remaining_subset)

class krypy.recycling.generators.RitzSmall(max_vectors=inf)

Bases: krypy.recycling.generators._RitzSubsetsGenerator

Successively returns the Ritz value of smallest magnitude.

generate (ritz, remaining_subset)
class krypy.recycling.generators._RitzSubsetsGenerator
    Abstract base class for the generation of subset generation.

    generate (ritz, remaining_subset)
        Returns a list of subsets with indices of Ritz vectors that are considered for deflation.

krypy.recycling.evaluators - evaluators for deflation vector candidates

class krypy.recycling.evaluators.RitzApproxKrylov (mode='extrapolate', tol=None, pseudospectra=False, bound_pseudo_kwargs=None, deflweight=1.0)

    Bases: krypy.recycling.evaluators._RitzSubsetEvaluator

    Evaluates a choice of Ritz vectors with a tailored approximate Krylov subspace method.

    Parameters
    • mode – (optional) determines how the number of iterations is estimated. Must be one
        of the following:
        • extrapolate (default): use the iteration count where the extrapolation of
          the smallest residual reduction over all steps drops below the tolerance.
        • direct: use the iteration count where the predicted residual bound drops below the
          tolerance. May result in severe underestimation if pseudospectra==False.
        • pseudospectra – (optional) should pseudospectra be computed for the given prob-
          lem? With pseudospectra=True, a prediction may not be possible due to unful-
          filled assumptions for the computation of the pseudospectral bound.
        • bound_pseudo_kwargs – (optional) a dictionary with arguments that are passed to
          bound_pseudo().
        • deflweight – (optional) see estimate_time(). Defaults to 1.

    evaluate (ritz, subset)

class krypy.recycling.evaluators.RitzApriori (Bound, tol=None, strategy='simple', deflweight=1.0)

    Bases: krypy.recycling.evaluators._RitzSubsetEvaluator

    Evaluates a choice of Ritz vectors with an a-priori bound for self-adjoint problems.

    Parameters
    • Bound – the a-priori bound which is used for estimating the convergence behavior.
    • tol – (optional) the tolerance for the stopping criterion, see _KrylovSolver. If
      None is provided (default), then the tolerance is retrieved from ritz.deflated_solver.tol
      in the call to evaluate().
    • strategy – (optional) the following strategies are available
        • simple: (default) uses the Ritz values that are complementary to the deflated ones for
          the evaluation of the bound.
        • intervals: uses intervals around the Ritz values that are considered with simple. The
          intervals incorporate possible changes in the operators.

    evaluate (ritz, subset)

class krypy.recycling.RecyclingCg (*args, **kwargs)

    Bases: krypy.recycling.linsys._RecyclingSolver

    Recycling preconditioned CG method.

    See _RecyclingSolver for the documentation of the available parameters.
class krypy.recycling.RecyclingMinres(*args, **kwargs)
Bases: krypy.recycling.linsys._RecyclingSolver
Recycling preconditioned MINRES method.
See _RecyclingSolver for the documentation of the available parameters.
class krypy.recycling.RecyclingGmres(*args, **kwargs)
Bases: krypy.recycling.linsys._RecyclingSolver
Recycling preconditioned GMRES method.
See _RecyclingSolver for the documentation of the available parameters.
class krypy.recycling.linsys._RecyclingSolver(DeflatedSolver, vector_factory=None)
Base class for recycling solvers.
Initialize recycling solver base.

Parameters

- **DeflatedSolver** – a deflated solver from deflation.
- **vector_factory** – (optional) An instance of a subclass of krypy.recycling.factories._DeflationVectorFactory that constructs deflation vectors for recycling. Defaults to None which means that no recycling is used.

Also the following strings are allowed as shortcuts:
- ‘RitzApproxKrylov’: uses the approximate Krylov subspace bound evaluator krypy.recycling.evaluators.RitzApproxKrylov.
- ‘RitzAprioriCg’: uses the CG $\kappa$-bound (krypy.utils.BoundCG) as an a priori bound with krypy.recycling.evaluators.RitzApriori.
- ‘RitzAprioriMinres’: uses the MINRES bound (krypy.utils.BoundMinres) as an a priori bound with krypy.recycling.evaluators.RitzApriori.

After a run of the provided DeflatedSolver via solve(), the resulting instance of the DeflatedSolver is available in the attribute last_solver.

last_solver = None
DeflatedSolver instance from last run of solve().
Instance of DeflatedSolver that resulted from the last call to solve(). Initialized with None before the first run.
solve(linear_system, vector_factory=None, *args, **kwargs)
Solve the given linear system with recycling.
The provided vector_factory determines which vectors are used for deflation.

Parameters

- **linear_system** – the LinearSystem that is about to be solved.
- **vector_factory** – (optional) see description in constructor.

All remaining arguments are passed to the DeflatedSolver.

Returns instance of DeflatedSolver which was used to obtain the approximate solution. The approximate solution is available under the attribute xk.

timings = None
Timings from last run of solve().
Timings of the vector factory runs and the actual solution processes.
2.1.4 krypy.utils - Krylov Subspace Utilities

The utils module provides helper functions for common tasks in the process of solving linear algebraic systems. Collection of standard functions.

This method provides functions like inner products, norms, ...

```
exception krypy.utils.ArgumentError
    Bases: Exception
    Raised when an argument is invalid.
    Analogue to ValueError which is not used here in order to be able to distinguish between built-in errors and krypy errors.

exception krypy.utils.AssumptionError
    Bases: Exception
    Raised when an assumption is not satisfied.
    Differs from ArgumentError in that all passed arguments are valid but computations reveal that assumptions are not satisfied and the result cannot be computed.

exception krypy.utils.ConvergenceError(msg, solver)
    Bases: Exception
    Raised when a method did not converge.
    The ConvergenceError holds a message describing the error and the attribute solver through which the last approximation and other relevant information can be retrieved.

exception krypy.utils.LinearOperatorError
    Bases: Exception
    Raised when a LinearOperator cannot be applied.

exception krypy.utils.InnerProductError
    Bases: Exception
    Raised when the inner product is indefinite.

exception krypy.utils_RuntimeError
    Bases: Exception
    Raised for errors that do not fit in any other exception.

class krypy.utils.Arnoldi(A, v, maxiter=None, ortho='mgs', M=None, Mv=None, Mv_norm=None, ip_B=None)
    Bases: object
    Arnoldi algorithm.
    Computes V and H such that \( AV_n = V_{n+1} H_n \). If the Krylov subspace becomes A-invariant then V and H are truncated such that \( AV_n = V_n H_n \).

Parameters

- **A** – a linear operator that can be used with scipy’s aslinearoperator with shape==\((N, N)\).
- **v** – the initial vector with shape==\((N, 1)\).
- **maxiter** – (optional) maximal number of iterations. Default: N.
- **ortho** – (optional) orthogonalization algorithm: may be one of
  - ’mgs’: modified Gram-Schmidt (default).
  - ’dmgs’: double Modified Gram-Schmidt.
  - ’lanczos’: Lanczos short recurrence.
– 'house': Householder.

- **M** – (optional) a self-adjoint and positive definite preconditioner. If M is provided, then also a second basis $P_n$ is constructed such that $V_n = MP_n$. This is of importance in preconditioned methods. M has to be None if ortho=='house' (see B).
- **ip_B** – (optional) defines the inner product to use. See inner().
ip_B has to be None if ortho=='house'. It’s unclear to me (andrenarchy), how a variant of the Householder QR algorithm can be used with a non-Euclidean inner product. Compare http://math.stackexchange.com/questions/433644/is-householder-orthogonalization-qr-practicable-for-non-euclidean-inner-products

advance()
Carry out one iteration of Arnoldi.

get()

get_last()

class krypy.utils.BoundCG(evals, exclude_zeros=False)
Bases: object

CG residual norm bound.

Computes the $\kappa$-bound for the CG error $A$-norm when the eigenvalues of the operator are given, see [LieS13].

Parameters
- **evals** – an array of eigenvalues $\lambda_1, \ldots, \lambda_N \in \mathbb{R}$. The eigenvalues will be sorted internally such that $0 = \lambda_1 = \ldots = \lambda_{t-1} < \lambda_t \leq \ldots \lambda_N$ for $t \in \mathbb{N}$.

- **steps** – (optional) the number of steps $k$ to compute the bound for. If steps is None (default), then $k = N$ is used.

Returns

array $[\eta_0, \ldots, \eta_k]$ with

$$
\eta_n = 2 \left( \frac{\sqrt{\kappa_{\text{eff}}}-1}{\sqrt{\kappa_{\text{eff}}}+1} \right)^n \quad \text{for} \quad n \in \{0, \ldots, k\}
$$

where $\kappa_{\text{eff}} = \frac{\lambda_N}{\lambda_1}$.

Initialize with array/list of eigenvalues or Intervals object.

eval_step(step)
Evaluate bound for given step.

get_step(tol)
Return step at which bound falls below tolerance.

class krypy.utils.BoundMinres(evals)
Bases: object

MINRES residual norm bound.

Computes a bound for the MINRES residual norm when the eigenvalues of the operator are given, see [Gre97].

Parameters
- **evals** – an array of eigenvalues $\lambda_1, \ldots, \lambda_N \in \mathbb{R}$. The eigenvalues will be sorted internally such that $\lambda_1 \leq \ldots \lambda_s < 0 = \lambda_{s+1} = \ldots = \lambda_{s+t-1} < \lambda_t \leq \ldots \lambda_N$ for $s, t \in \mathbb{N}$ and $s < t$.

- **steps** – (optional) the number of steps $k$ to compute the bound for. If steps is None (default), then $k = N$ is used.
Returns

array \{\eta_0, \ldots, \eta_k\} with

\eta_n = 2 \left( \sqrt{\frac{\lambda_1 \lambda_{N}}{\lambda_1 \lambda_{N} + \lambda_s \lambda_t}} \right)^{\frac{2}{n}} \quad \text{for } n \in \{0, \ldots, k\}

if \ s > 0. \ If \ s = 0, \ i.e., \ if \ the \ eigenvalues \ are \ non-negative, \ then \ the \ result \ of bound_cg() \ is \ returned.

Initialize with array/list of eigenvalues or Intervals object.

static __new__(evals)

Use BoundCG if all eigenvalues are non-negative.

eval_step(step)

Evaluate bound for given step.

get_step(tol)

Return step at which bound falls below tolerance.

exception krypy.utils.ConvergenceError(msg, solver)

Bases: Exception

Raised when a method did not converge.

The ConvergenceError holds a message describing the error and the attribute solver through which the last approximation and other relevant information can be retrieved.

class krypy.utils.Givens(x)

Bases: object

Compute Givens rotation for provided vector x.

Computes Givens rotation \( G = \begin{bmatrix} c & s \\ -\bar{s} & c \end{bmatrix} \) such that \( Gx = \begin{bmatrix} r \\ 0 \end{bmatrix} \).

apply(x)

Apply Givens rotation to vector x.

class krypy.utils.House(x)

Bases: object

Compute Householder transformation for given vector.

Initialize Householder transformation \( H \) such that \( Hx = \alpha \|x\| e_1 \) with \( |\alpha| = 1 \)


apply(x)

Apply Householder transformation to vector x.

Applies the Householder transformation efficiently to the given vector.

matrix()

Build matrix representation of Householder transformation.

Builds the matrix representation \( H = I - \beta vv^* \).

Use with care! This routine may be helpful for testing purposes but should not be used in production codes for high dimensions since the resulting matrix is dense.

class krypy.utils.IdentityLinearOperator(shape)

Bases: krypy.utils.LinearOperator

class krypy.utils.LinearOperator(shape, dtype=None, dot=None, dot_adj=None)

Bases: object

Linear operator.
Is partly based on the LinearOperator from scipy (BSD License).

```python
__add__(X)
__mul__(X)
__neg__()
__pow__(X)
__repr__()
__rmul__(X)
__sub__(X)
adj
dot(X)
dot_adj(X)
```

class krypy.utils.MatrixLinearOperator(A):
    Bases: krypy.utils.LinearOperator
    __repr__()

class krypy.utils.NormalizedRootsPolynomial(roots):
    Bases: object
    A polynomial with specified roots and p(0)=1.
    Represents the polynomial
    \[ p(\lambda) = \prod_{i=1}^{n} \left( 1 - \frac{\lambda}{\theta_i} \right). \]
    Parameters roots - array with roots \( \theta_1, \ldots, \theta_n \) of the polynomial and roots.shape==(n,).
    __call__(points)
    Evaluate polynomial at given points.
    Parameters points - a point \( x \) or array of points \( x_1, \ldots, x_m \) with points.shape==(m,).
    Returns \( p(x) \) or array of shape \( (m,) \) with \( p(x_1), \ldots, p(x_m) \).

    minmax_candidates()
    Get points where derivative is zero.
    Useful for computing the extrema of the polynomial over an interval if the polynomial has real roots.
    In this case, the maximum is attained for one of the interval endpoints or a point from the result of this function that is contained in the interval.

class krypy.utils.Projection(X, Y=None, ip_B=None, orthogonalize=True, iterations=2):
    Bases: object
    Generic projection.
    This class can represent any projection (orthogonal and oblique) on a N-dimensional Hilbert space. A projection is a linear operator \( P \) with \( P^2 = P \). A projection is uniquely defined by its range \( \mathcal{V} := \text{range}(P) \) and its kernel \( \mathcal{W} := \ker(P) \); this projection is called \( P_{\mathcal{V}^\perp} \).
    Let \( X \) and \( Y \) be two full rank arrays with shape\(=(N, k)\) and let \( \mathcal{X} \oplus \mathcal{Y}^\perp = \mathbb{C}^N \) where \( \mathcal{X} := \text{colspan}(X) \) and \( \mathcal{Y} := \text{colspan}(Y) \). Then this class constructs the projection \( P_{\mathcal{X} \oplus \mathcal{Y}^\perp} \). The requirement \( \mathcal{X} \oplus \mathcal{Y}^\perp = \mathbb{C}^N \) is equivalent to \langle X, Y \rangle \text{ranging being nonsingular}.
    Parameters
    * \( X \) - array with shape\(=(N, k)\) and \( \text{rank}(X) = k \).
This projection class makes use of the round-off error analysis of oblique projections in the work of Stewart [Ste11] and implements the algorithms that are considered as the most stable ones (e.g., the XQRY representation in [Ste11]).

**apply** *(a, return_Ya=False)*

Apply the projection to an array.

The computation is carried out without explicitly forming the matrix corresponding to the projection (which would be an array with shape==(N,N)).

See also _apply().

**apply_adj** *(a)*

**apply_complement** *(a, return_Ya=False)*

Apply the complementary projection to an array.

Parameters

- **z** – array with shape==(N,m).

Returns

\( P_{\perp, Y} z = z - P_{X, Y} z \).

**apply_complement_adj** *(a)*

**matrix** ()

Builds matrix representation of projection.

Builds the matrix representation \( P = X(Y, X)^{-1}(Y, I_N) \).

**operator** ()

Get a LinearOperator corresponding to apply().

**operator_complement** ()

Get a LinearOperator corresponding to apply_complement().

**class** krypy.utils.Timer

Bases: list

Measure execution time of multiple code blocks with with.

Example:

```python
from krypy import Timer
t = Timer()
with t:
    print('time me!')
    print('don\t time me!')
with t:
    print('time me, too!')
print(t)
```
KryPy Documentation, Release 2.1.4

Result:

time me!
don’t time me!
time me, too!

[6.389617919921875e-05, 6.008148193359375e-05]

```
__enter__()
__exit__(a, b, c)
```

**krypy.utils.angles**(F, G, ip_B=None, compute_vectors=False)

Principal angles between two subspaces.

This algorithm is based on algorithm 6.2 in Knyazev, Argentati. *Principal angles between subspaces in an A-based scalar product: algorithms and perturbation estimates*. 2002. This algorithm can also handle small angles (in contrast to the naive cosine-based svd algorithm).

**Parameters**

- **F** – array with shape==(N,k).
- **G** – array with shape==(N,l).
- **ip_B** – (optional) angles are computed with respect to this inner product. See `inner()`.
- **compute_vectors** – (optional) if set to False then only the angles are returned (default). If set to True then also the principal vectors are returned.

**Returns**

- **theta** if compute_vectors==False
- **theta, U, V** if compute_vectors==True

where

- **theta** is the array with shape==(max(k,l),) containing the principal angles $0 \leq \theta_1 \leq \ldots \leq \theta_{\max\{k,l\}} \leq \frac{\pi}{2}$.
- **U** are the principal vectors from F with $\langle U, U \rangle = I_k$.
- **V** are the principal vectors from G with $\langle V, V \rangle = I_l$.

The principal angles and vectors fulfill the relation $\langle U, V \rangle = \begin{bmatrix} \cos(\Theta) & 0_{m,l-m} \\ 0_{k-m,m} & 0_{k-m,l-m} \end{bmatrix}$ where $m = \min\{k,l\}$ and $\cos(\Theta) = \text{diag}(\cos(\theta_1), \ldots, \cos(\theta_m))$. Furthermore, $\theta_{m+1} = \ldots = \theta_{\max\{k,l\}} = \frac{\pi}{2}$.

```
krypy.utils.arnoldi(*args, **kwargs)
krypy.utils.arnoldi_res(A, V, H, ip_B=None)
```

Measure Arnoldi residual.

**Parameters**

- **A** – a linear operator that can be used with scipy’s aslinearoperator with shape==(N,N).
- **V** – Arnoldi basis matrix with shape==(N,n).
- **H** – Hessenberg matrix: either $H_{n-1}$ with shape==(n,n-1) or $H_n$ with shape==(n,n) (if the Arnoldi basis spans an A-invariant subspace).
- **ip_B** – (optional) the inner product to use, see `inner()`.

**Returns** either $\|AV_n-1 - V_nH_{n-1}\|$ or $\|AV_n - V_nH_n\|$ (in the invariant case).

```
krypy.utils.arnoldi_projected(H, P, k, ortho='mgs')
```

Compute (perturbed) Arnoldi relation for projected operator.
Assume that you have computed an Arnoldi relation

\[ AV_n = V_{n+1} H_n \]

where \( V_{n+1} \in \mathbb{C}^{N,n+1} \) has orthogonal columns (with respect to an inner product \( \langle \cdot, \cdot \rangle \)) and \( H_n \in \mathbb{C}^{n+1,n} \) is an extended upper Hessenberg matrix.

For \( k < n \) you choose full rank matrices \( X \in \mathbb{C}^{n-1,k} \) and \( Y \in \mathbb{C}^{n,k} \) and define \( \tilde{X} := AV_n X = V_n H_{n-1} X \) and \( \tilde{Y} := V_n Y \) such that \( \langle \tilde{Y}, \tilde{X} \rangle = Y^* H_{n-1} X \) is invertible. Then the projections \( P \) and \( \tilde{P} \) characterized by

\[ \tilde{P} x = x - \tilde{X} \langle \tilde{Y}, \tilde{X} \rangle^{-1} \langle \tilde{Y}, x \rangle \]

\[ P = I - H_{n-1} X (Y^* H_{n-1} X)^{-1} Y^* \]

are well defined and \( \tilde{P} V_{n+1} = [V_n P, v_{n+1}] \) holds.

This method computes for \( i < n - k \) the Arnoldi relation

\[ (\tilde{P} A + E_i) W_i = W_{i+1} G_i \]

where \( W_{i+1} = V_n U_{i+1} \) has orthogonal columns with respect to \( \langle \cdot, \cdot \rangle \), \( G_i \) is an extended upper Hessenberg matrix and \( E_i x = v_{n+1} F_i (W_i, x) \) with \( F_i = [f_1, \ldots, f_i] \in \mathbb{C}^{i+1,i} \).

The perturbed Arnoldi relation can also be generated with the operator \( P_{V_n} \tilde{P} A \):

\[ P_{V_n} \tilde{P} A W_i = W_{i+1} G_i \]

In a sense the perturbed Arnoldi relation is the best prediction for the behavior of the Krylov subspace \( K_i(\tilde{P} A, P_{V_n}) \) that can be generated only with the data from \( K_{n+1}(A, v_1) \) and without carrying out further matrix-vector multiplications with \( A \).

**Parameters**

- \( H \) – the extended upper Hessenberg matrix \( H_n \) with \( \text{shape}==(n+1,n) \).
- \( P \) – the projection \( P : \mathbb{C}^n \rightarrow \mathbb{C}^n \) (has to be compatible with \( \text{get_linearoperator()} \)).
- \( k \) – the dimension of the null space of \( P \).

**Returns**

- \( U, G, F \) where
  - \( U \) is the coefficient matrix \( U_{i+1} \) with \( \text{shape}==(n,i+1) \).
  - \( G \) is the extended upper Hessenberg matrix \( G_i \) with \( \text{shape}==(i+1,i) \).
  - \( F \) is the error matrix \( F_i \) with \( \text{shape}==(1,i) \).

**Examples**

```python
krypy.utils.bound_perturbed_gmres(pseudo, p, epsilon, deltas)
```

Computes GMRES perturbation bound based on pseudospectrum

**Examples**

```python
krypy.utils.gap(lambda, sigma, mode='individual')
```

Compute spectral gap.

Useful for eigenvalue/eigenvector bounds. Computes the gap \( \delta \geq 0 \) between two sets of real numbers \( \Lambda \) and \( \Sigma \). The gap can be computed in several ways and may not exist, see the \( \text{mode} \) parameter.

**Parameters**

- \( \Lambda \) – a non-empty set \( \Lambda = \{\lambda_1, \ldots, \lambda_n\} \) given as a single real number or a list or \( \text{numpy.array} \) with real numbers.
- \( \Sigma \) – a non-empty set \( \Sigma = \{\sigma_1, \ldots, \sigma_m\} \). See \( \Lambda \).
- \( \text{mode} \) – (optional). Defines how the gap should be computed. May be one of
- 'individual' (default): $\delta = \min_{i \in \{1, \ldots, n\}} |\lambda_i - \sigma_j|$. With this mode, the gap is always defined.

- 'interval': determine the maximal $\delta$ such that $\Sigma \subset \mathbb{R} \setminus [\min_{\lambda \in \Lambda} \lambda - \delta, \max_{\lambda \in \Lambda} \lambda + \delta]$. If the gap does not exist, None is returned.

Returns $\delta$ or None.

krypy.utils.get_linearoperator(shape, A, timer=None)
Enhances aslinearoperator if A is None.

krypy.utils.hegedus(A, b, x0, M=None, Ml=None, ip_B=None)
Rescale initial guess appropriately (Hegedüs trick).

The Hegedüs trick rescales the initial guess to $\gamma_{\min}x_0$ such that

$$
\|r_0\|_{M^{-1}} = \|MM_l(b - A\gamma_{\min}x_0)\|_{M^{-1}} = \min_{\gamma \in \mathbb{C}} \|MM_l(b - A\gamma x_0)\|_{M^{-1}} \leq \|MM_l b\|_{M^{-1}}.
$$

This is achieved by $\gamma_{\min} = \frac{(z, MM_l b)_{M^{-1}}}{\|z\|_{M^{-1}}}$ for $z = MM_l Ax_0$ because then $r_0 = P_zb$. (Note that the right hand side of formula (5.8.16) in [LieS13] has to be complex conjugated.)

The parameters are the parameters you want to pass to gmres, minres or cg.

Returns the adapted initial guess with the above property.

krypy.utils.inner(X, Y, ip_B=None)
Euclidean and non-Euclidean inner product.

numpy.vdot only works for vectors and numpy.dot does not use the conjugate transpose.

Parameters

- X – numpy array with shape==(N,m)
- Y – numpy array with shape==(N,n)
- ip_B – (optional) May be one of the following
  - None: Euclidean inner product.
  - a self-adjoint and positive definite operator $B$ (as numpy.array or LinearOperator). Then $X^*BY$ is returned.
  - a callable which takes 2 arguments X and Y and returns $\langle X, Y \rangle$.

Caution: a callable should only be used if necessary. The choice potentially has an impact on the round-off behavior, e.g. of projections.

Returns numpy array $(X,Y)$ with shape==(m,n).

krypy.utils.ip_euclid(X, Y)
Euclidean inner product.

numpy.vdot only works for vectors and numpy.dot does not use the conjugate transpose.

Parameters

- X – numpy array with shape==(N,m)
- Y – numpy array with shape==(N,n)

Returns numpy array $X^*Y$ with shape==(m,n).

krypy.utils.norm(x, y=None, ip_B=None)
Compute norm (Euclidean and non-Euclidean).

Parameters

- x – a 2-dimensional numpy.array.
- y – a 2-dimensional numpy.array.
• ip_B – see inner().

Compute $\sqrt{\langle x, y \rangle}$ where the inner product is defined via ip_B.

```python
krypy.utils.norm_MMr(M, Ml, A, Mr, b, x0, yk, inner_product=<function ip_euclid>)
```

Compute the norm^2 w.r.t. to a given scalar product.

```python
krypy.utils.norm_squared(x, Mx=None, inner_product=<function ip_euclid>)
```

Compute the norm^2 w.r.t. to a given scalar product.

```python
krypy.utils.orthonormality(V, ip_B=None)
```

Measure orthonormality of given basis.

Parameters

- V – a matrix $V = [v_1, \ldots, v_n]$ with shape==(N,n).
- ip_B – (optional) the inner product to use, see inner().

Returns $\|I_n - \langle V, V \rangle\|_2$.

```python
krypy.utils.qr(X, ip_B=None, reorthos=1)
```

QR factorization with customizable inner product.

Parameters

- X – array with shape==(N,k)
- ip_B – (optional) inner product, see inner().
- reorthos – (optional) number of reorthogonalizations. Defaults to 1 (i.e. 2 runs of modified Gram-Schmidt) which should be enough in most cases (TODO: add reference).

Returns Q, R where $X = QR$ with $\langle Q, Q \rangle = I_k$ and R upper triangular.

```python
krypy.utils.ritz(H, V=None, hermitian=False, type='ritz')
```

Compute several kinds of Ritz pairs from an Arnoldi/Lanczos relation.

This function computes Ritz, harmonic Ritz or improved harmonic Ritz values and vectors with respect to the Krylov subspace $K_n(A, v)$ from the extended Hessenberg matrix $H_n$ generated with n iterations the Arnoldi algorithm applied to A and v.

Parameters

- H – Hessenberg matrix from Arnoldi/Lanczos algorithm.
- V – (optional) Arnoldi/Lanczos vectors, $V \in \mathbb{C}^{N,n+1}$. If provided, the Ritz vectors are also returned. The Arnoldi vectors have to form an orthonormal basis with respect to an inner product.

Caution: if you are using the Lanczos or Gram-Schmidt Arnoldi algorithm without reorthogonalization, then the orthonormality of the basis is usually lost. For accurate results it is advisable to use the Householder Arnoldi (ortho='house') or modified Gram-Schmidt with reorthogonalization (ortho='dmgs').

- hermitian – (optional) if set to True the matrix $H_n$ must be Hermitian. A Hermitian matrix $H_n$ allows for faster and often more accurate computation of Ritz pairs.
- type – (optional) type of Ritz pairs, may be one of 'ritz', 'harmonic' or 'harmonic_like'. Two choices of Ritz pairs fit in the following description:

Given two n-dimensional subspaces $X, Y \subseteq \mathbb{C}^N$, find a basis $z_1, \ldots, z_n$ of $X$ and $\theta_1, \ldots, \theta_n \in \mathbb{C}$ such that $Az_i - \theta_i z_i \perp Y$ for all $i \in \{1, \ldots, n\}$.

In this setting the choices are

- 'ritz': regular Ritz pairs, i.e. $X = Y = K_n(A, v)$.
- 'harmonic': harmonic Ritz pairs, i.e. $X = K_n(A, v)$ and $Y = AK_n(A, v)$. 

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– `'harmonic_improved'`: the returned vectors U (and V, if requested) are the same as with type='harmonic'. The theta array contains the improved Ritz values \( \theta_i = u_i^* H_n u_i \), cf. section 2 in Morgan, Zeng. *Harmonic Projection Methods for Large Non-symmetric Eigenvalue Problems*. 1998. It can be shown that the residual norm of improved Ritz pairs is always less than or equal to the residual norm of the harmonic Ritz pairs. However, the improved Ritz pairs do not fit into the framework above since the orthogonality condition is lost.

**Returns**

- If V is not None then theta, U, resnorm, Z is returned.
- If V is None then theta, U, resnorm is returned.

Where

- theta are the Ritz values \([\theta_1, \ldots, \theta_n]\).
- U are the coefficients of the Ritz vectors in the Arnoldi basis, i.e. \( z_i = V u_i \) where \( u_i \) is the i-th column of U.
- resnorm is a residual norm vector.
- Z are the actual Ritz vectors, i.e. \( Z = \text{dot}(V, U) \).

```python
krypy.utils.shape_vec(x)
    Take a (n,) ndarray and return it as (n,1) ndarray.
```

```python
krypy.utils.shape_vecs(*args)
    Reshape all ndarrays with shape==(n,) to shape==(n,1).
    Recognizes ndarrays and ignores all others.
```

```python
krypy.utils.strakos(n, l_min=0.1, l_max=100, rho=0.9)
    Return the Strakoš matrix.
    See [Str92].
```

### 2.1.5 Subpackages

**tests Package**

**tests Module**

```python
krypy.tests.test_linsys.check_solver(sol, solver, ls, params)
krypy.tests.test_linsys.dictpick(d)
krypy.tests.test_linsys.dictproduct(d)
    enhance itertools product to process values of dicts
    example: d = {'a':[1,2], 'b':[3,4]} then list(dictproduct(d)) == [{'a':1,'b':3}, {'a':1,'b':4}, {'a':2,'b':3}, {'a':2,'b':4}]
krypy.tests.test_linsys.linear_systems_generator(A, **ls_kwargs)
krypy.tests.test_linsys.run_solver(solver, ls, params)
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2.2 Bibliography
3.1 Installation

KryPy can be installed easily with the Python package installer by issuing `pip install krypy`. Alternatively, it can be installed by downloading the source from KryPy’s github page and then running `python setup.py install`.

3.2 Solve a linear system

The following code uses MINRES to solves a linear system with an indefinite diagonal matrix:

```python
from numpy import diag, linspace, ones, eye
from krypy.linalg import LinearSystem, Minres

# construct the linear system
A = diag(linspace(1, 2, 20))
A[0, 0] = -1e-5
b = ones(20)
linear_system = LinearSystem(A, b, self_adjoint=True)

# solve the linear system (approximate solution is solver.xk)
solver = Minres(linear_system)
```

3.3 Deflation

The vector $e_1$ can be used as a deflation vector to get rid of the small negative eigenvalue $-10^{-5}$:

```python
from krypy.deflation import DeflatedMinres
dsolver = DeflatedMinres(linear_system, U=eye(20, 1))
```

3.4 Recycling

The deflation subspace can also be determined automatically with a recycling strategy. Just for illustration, the same linear system is solved twice in the following code:

```python
from krypy.recycling import RecyclingMinres

# get recycling solver with approximate Krylov subspace strategy
rminres = RecyclingMinres(vector_factory='RitzApproxKrylov')
```
```python
# solve twice
rresolver1 = rminres.solve(linear_system)
rresolver2 = rminres.solve(linear_system)
```

The convergence histories can be plotted by

```python
from matplotlib.pyplot import semilogy, show, legend
semilogy(solver.resnorms, label='original')
semilogy(dsolver.resnorms, label='exact deflation', ls='dotted')
semilogy(rsolver2.resnorms, label='automatic recycling', ls='dashed')
legend()
show()
```

which results in the following figure.
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