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1.1 fenics_shells.analytical package

1.1.1 Submodules

1.1.2 fenics_shells.analytical.lovadina_clamped module

Analytical solution for clamped Reissner-Mindlin plate problem from Lovadina et al.

1.1.3 fenics_shells.analytical.simply_supported module

Analytical solution for simply-supported Reissner-Mindlin square plate under a uniform transverse load.

1.1.4 fenics_shells.analytical.vonkarman_heated module

Analytical solution for elliptic orthotropic von Karman plate with lenticular thickness subject to a uniform field of inelastic curvatures.

\[
\text{fenics_shells.analytical.vonkarman_heated.analytical_solution}(A_i, D_i, a_{rad}, b_{rad})
\]
FEniCS-Shells, Release 2018.1.0

1.1.5 Module contents

1.2 fenics_shells.common package

1.2.1 Submodules

1.2.2 fenics_shells.common.constitutive_models module

fenics_shells.common.constitutive_models.

psi_M(k, **kwargs)

Returns bending moment energy density calculated from the curvature k using:

Isotropic case: .. math:: D = \frac{E*t^3}{24(1 - nu^2)} W_m(k, \ldots) = D*((1 - nu)*tr(k**2) + nu*(tr(k))*2)

Parameters

• k – Curvature, typically UFL form with shape (2,2) (tensor).

• **kwargs – Isotropic case: E: Young’s modulus, Constant or Expression. nu: Poisson’s ratio, Constant or Expression. t: Thickness, Constant or Expression.

Returns UFL form of bending stress tensor with shape (2,2) (tensor).

fenics_shells.common.constitutive_models.

psi_N(e, **kwargs)

Returns membrane energy density calculated from e using:

Isotropic case: .. math:: B = \frac{E*t}{2(1 - nu^2)} N(e, \ldots) = B(1 - nu)e + nu \text{tr}(e)I

Parameters

• e – Membrane strain, typically UFL form with shape (2,2) (tensor).

• **kwargs – Isotropic case: E: Young’s modulus, Constant or Expression. nu: Poisson’s ratio, Constant or Expression. t: Thickness, Constant or Expression.

Returns UFL form of membrane stress tensor with shape (2,2) (tensor).

fenics_shells.common.constitutive_models.

strain_from_voigt(e_voigt)

Inverse operation of strain_to_voigt.

Parameters

• sigma_voigt – UFL form with shape (3,1) corresponding to the strain

• in Voigt format (pseudo-vector)

Returns a symmetric stress tensor, typically UFL form with shape (2,2)

fenics_shells.common.constitutive_models.

strain_to_voigt(e)

Returns the pseudo-vector in the Voigt notation associate to a 2x2 symmetric strain tensor, according to the following rule (see e.g. https://en.wikipedia.org/wiki/Voigt_notation),

\[ e = \begin{bmatrix} e_{00} & e_{01} \\ e_{01} & e_{11} \end{bmatrix} \rightarrow e_{\text{voigt}} = \begin{bmatrix} e_{00} & e_{11} & 2e_{01} \end{bmatrix} \]

Parameters e – a symmetric 2x2 strain tensor, typically UFL form with shape (2,2)

Returns a UFL form with shape (3,1) corresponding to the input tensor in Voigt notation.

fenics_shells.common.constitutive_models.

stress_from_voigt(sigma_voigt)

Inverse operation of stress_to_voigt.
Parameters

- **sigma_voigt** – UFL form with shape (3,1) corresponding to the stress in Voigt format. (pseudo-vector)

Returns a symmetric stress tensor, typically UFL form with shape (2,2)

```
fenics_shells.common.constitutive_models.stress_to_voigt(sigma)
```

Returns the pseudo-vector in the Voigt notation associate to a 2x2 symmetric stress tensor, according to the following rule (see e.g. https://en.wikipedia.org/wiki/Voigt_notation),

\[
\sigma = \begin{bmatrix}
\sigma_{00} & \sigma_{01} \\
\sigma_{01} & \sigma_{11}
\end{bmatrix} \rightarrow \sigma_{\text{voigt}} = \begin{bmatrix}
\sigma_{00} & \sigma_{11} \\
\sigma_{01} & \sigma_{01}
\end{bmatrix}
\]

Parameters

- **sigma** – a symmetric 2x2 stress tensor, typically UFL form with shape (2,2)

Returns a UFL form with shape (3,1) corresponding to the input tensor in Voigt notation.

1.2.3 fenics_shells.common.energy module

```
fenics_shells.common.energy.membrane_bending_energy(e, k, A, D, B)
```

Return the coupled membrane-bending energy for a plate model.

Parameters

- **e** – Membrane strains, UFL or DOLFIN Function of rank (2, 2) (tensor).
- **k** – Curvature, UFL or DOLFIN Function of rank (2, 2) (tensor).
- **A** – Membrane stresses.
- **D** – Bending stresses.
- **B** – Coupled membrane-bending stresses.

```
fenics_shells.common.energy.membrane_energy(e, N)
```

Return internal membrane energy for a plate model.

Parameters

- **e** – Membrane strains, UFL or DOLFIN Function of rank (2, 2) (tensor).
- **N** – Membrane stress, UFL or DOLFIN Function of rank (2, 2) (tensor).

Returns UFL form of internal elastic membrane energy for a plate model.

1.2.4 fenics_shells.common.kinematics module

```
fenics_shells.common.kinematics.F(u)
```

Return deformation gradient tensor for non-linear plate model.

Deformation gradient of 2-dimensional manifold embedded in 3-dimensional space.

\[
F = I + \nabla u
\]

Parameters **u** – displacement field, typically UFL (3,1) coefficient.
Returns a UFL coefficient with shape (3,2)

\[ e = \frac{1}{2} (\nabla u + \nabla u^T) \]

Parameters  
- \( u \) – membrane displacement field, typically UFL (2,1) coefficient.

Returns a UFL form with shape (2,2)

\[ k = \frac{1}{2} (\nabla \theta + \nabla \theta^T) \]

Parameters  
- \( \theta \) – rotation field, typically UFL (2,1) form or a dolfin Function

1.2.5 fenics_shells.common.laminates module

fenics_shells.common.laminates.ABD(\( E_1, E_2, G_{12}, nu_{12}, hs, thetas \))

Return the stiffness matrix of a kirchhoff-love model of a laminate obtained by stacking \( n \) orthotropic laminae with possibly different thinknesses and orientations (see Reddy 1997, eqn 1.3.71).

It assumes a plane-stress state.

Parameters

- \( E_1 \) – The Young modulus in the material direction 1.
- \( E_2 \) – The Young modulus in the material direction 2.
- \( G_{12} \) – The in-plane shear modulus.
- \( nu_{12} \) – The in-plane Poisson ratio.
- \( hs \) – a list with length \( n \) with the thicknesses of the layers (from top to bottom).
- \( thetas \) – a list with the \( n \) orientations (in radians) of the layers (from top to bottom).

Returns a symmetric 3x3 ufl matrix giving the membrane stiffness in Voigt notation. B: a symmetric 3x3 ufl matrix giving the membrane/bending coupling stiffness in Voigt notation. D: a symmetric 3x3 ufl matrix giving the bending stiffness in Voigt notation.

Return type  

A

fenics_shells.common.laminates.F(\( G_{13}, G_{23}, hs, thetas \))

Return the shear stiffness matrix of a Reissner-Midlin model of a laminate obtained by stacking \( n \) orthotropic laminae with possibly different thinknesses and orientations (See Reddy 1997, eqn 3.4.18)

It assumes a plane-stress state.

Parameters

- \( G_{13} \) – The transverse shear modulus between the material directions 1-3.
- \( G_{23} \) – The transverse shear modulus between the material directions 2-3.
- \( hs \) – a list with length \( n \) with the thicknesses of the layers (from top to bottom).
• **theta** – a list with the n orientations (in radians) of the layers (from top to bottom).

**Returns** a symmetric 2x2 ufl matrix giving the shear stiffness in Voigt notation.

**Return type** `F`

`fenics_shells.common.laminates.NM_T(E1, E2, G12, nu12, hs, thetas, DeltaT_0, DeltaT_1=0.0, alpha1=1.0, alpha2=1.0)`

Return the thermal stress and moment resultant of a Kirchhoff-Love model of a laminate obtained by stacking n orthotropic laminae with possibly different thicknesses and orientations. It assumes a plane-stress state and a temperature distribution in the form

\[\Delta(z) = \Delta T_0 + z \cdot \Delta T_1\]

**Parameters**

- **E1** – The Young modulus in the material direction 1.
- **E2** – The Young modulus in the material direction 2.
- **G12** – The in-plane shear modulus.
- **nu12** – The in-plane Poisson ratio.
- **hs** – a list with length n with the thicknesses of the layers (from top to bottom).
- **theta** – a list with the n orientations (in radians) of the layers (from top to bottom).
- **alpha1** – Expansion coefficient in the material direction 1.
- **alpha2** – Expansion coefficient in the material direction 2.
- **DeltaT_0** – Average temperature field.
- **DeltaT_1** – Gradient of the temperature field.

**Returns** a 3x1 ufl vector giving the membrane inelastic stress. M_T: a 3x1 ufl vector giving the bending inelastic stress.

**Return type** `N_T`

`fenics_shells.common.laminates.rotated_lamina_expansion_inplane(alpha11, alpha22, theta)`

Return the in-plane expansion matrix of an orthotropic layer in a reference rotated by an angle theta wrt to the material one. It assumes Voigt notation and plane stress state. (See Reddy 1997, eqn 1.3.71)

**Parameters**

- **alpha11** – Expansion coefficient in the material direction 1.
- **alpha22** – Expansion coefficient in the material direction 2.
- **theta** – The rotation angle from the material to the desired reference system.

**Returns** a 3x1 ufl vector giving the expansion matrix in voigt notation.

**Return type** `alpha_theta`

`fenics_shells.common.laminates.rotated_lamina_stiffness_inplane(E1, E2, G12, nu12, theta)`

Return the in-plane stiffness matrix of an orthotropic layer in a reference rotated by an angle theta wrt to the material one. It assumes Voigt notation and plane stress state. (See Reddy 1997, eqn 1.3.71)

**Parameters**

- **E1** – The Young modulus in the material direction 1.
- **E2** – The Young modulus in the material direction 2.
• **G23** – The in-plane shear modulus
• **nu12** – The in-plane Poisson ratio
• **theta** – The rotation angle from the material to the desired reference system

**Returns** a 3x3 symmetric ufl matrix giving the stiffness matrix

**Return type** `Q_theta`

```python
def rotated_lamina_stiffness_shear(G13, G23, theta, kappa=0.8333333333333334):
    return fenics_shells.common.laminates.rotated_lamina_stiffness_shear(G13, G23, theta, kappa)
```

Return the shear stiffness matrix of an orthotropic layer in a reference rotated by an angle theta wrt to the material one. It assumes Voigt notation and plane stress state (see Reddy 1997, eqn 3.4.18).

**Parameters**

• **G12** – The transverse shear modulus between the material directions 1-2.
• **G13** – The transverse shear modulus between the material directions 1-3.
• **kappa** – The shear correction factor.

**Returns** a 3x3 symmetric ufl matrix giving the stiffness matrix.

**Return type** `Q_shear_theta`

```python
def z_coordinates(hs):
    return fenics_shells.common.laminates.z_coordinates(hs)
```

Return a list with the thickness coordinate of the top surface of each layer taking the midplane as z = 0.

**Parameters** `hs` – a list giving the thicknesses of each layer ordered from bottom (layer 0) to top (layer n-1).

**Returns**

a list of coordinate of the top surface of each layer ordered from bottom (layer 0) to top (layer n-1)

**Return type** `z`

### 1.2.6 Module contents

### 1.3 fenics_shells.fem package

#### 1.3.1 Submodules

#### 1.3.2 fenics_shells.fem.CDG module

```python
def cdg_energy(theta, M, stabilization, mesh, bcs_theta=None, dS=<Mock id='140294208557576'>):
    return fenics_shells.fem.CDG.cdg_energy(theta, M, stabilization, mesh, bcs_theta, dS)
```

Return the continuous/discontinuous terms for a fourth-order plate model.

\[
\pi_{cdg} = -\partial_n w \cdot M_n(w) + \frac{1}{2} \alpha |\partial_n w|^2
\]

**Parameters**

• **theta** – Rotations, UFL or DOLFIN Function of rank (2,) (vector).
• **M** – UFL form of bending moment tensor of rank (2,2) (tensor).
• **stabilization** – a constant or ulf expression providing the stabilization parameter of the continuous/discontinuous formulation. This should be an estimation of the norm of the bending stiffness

• **mesh** – DOLFIN mesh.

• **bcsttheta** *(Optional)* – list of dolfin.DirichletBC for the rotations theta. Defaults to None.

• **dS** – (Optional). Measure on interior facets. Defaults to dolfin.dS.

Returns a dolfin.Form associated with the continuous/discontinuous formulation.

The Kirchhoff-Love plate model is a fourth-order PDE, giving rise to a weak form with solution in Sobolev space $H^2(\Omega)$. Because FEniCS does not currently include support for $H^2(\Omega)$ conforming elements we implement a hybrid continuous/discontinuous approach, allowing the use of Lagrangian elements with reduced regularity requirements.


fenics_shells.fem.CDG.cdg_stabilization($E, t$)

Returns the stabilization parameter as the norm of the bending stiffness matrix.

Parameters

• $E$ – Young’s modulus, Constant or Expression.

• $t$ – Thickness, Constant or Expression.

Returns a dolfin.Coefficient providing the stabilization parameter of the continuous/discontinuous formulation.

1.3.3 fenics_shells.fem.assembling module

fenics_shells.fem.assembling.assemble(*args, **kwargs)

Pass-through for dolfin.assemble and fenics_shells.projected_assemble.

If the first argument is an instance of ProjectedFunctionSpace it will call fenics_shells.projected_assemble, otherwise it will pass through to dolfin.assemble.

fenics_shells.fem.assembling.projected_assemble($U_P, a, L, bcs=None, A=None, b=None, is_interpolation=False, a_is_symmetric=False, form_compiler_parameters=None, add_values=False, finalize_tensor=True, keep_diagonal=False, backend=None$)

1.3.4 fenics_shells.fem.solving module

fenics_shells.fem.solving.project($u, V$)

1.3. fenics_shells.fem package
Given a Function on a projected space \( u_p \in U_P \) and a function in the full space \( u_f \in U_F \) such that \( U_P \subset U_F \), reconstruct the variable \( u_f \) on the full space via direct copy of the matching subfunctions shared between \( U_F \) and \( U_P \) and the local solution of the original problem \( a = L \).

**Parameters**

- \( u_f \) – DOLFIN Function on FullFunctionSpace.
- \( u_p \) – DOLFIN Function on ProjectedFunctionSpace.
- \( a \) – Bilinear form.
- \( L \) – Bilinear form.

**Returns** DOLFIN Function on FullFunctionSpace.

**Return type** \( u_f \)

### 1.3.5 Module contents

**fenics_shells.fem.projected_assemble**

\[
\text{projected_assemble}(U_P, a, L, bc=None, A=None, b=None, 
  is_interpolation=False, a_is_symmetric=False, 
  form_compiler_parameters=None, add_values=False, 
  finalize_tensor=True, keep_diagonal=False, backend=None)
\]

**fenics_shells.fem.project**

\[
\text{project}(u, V)
\]

**fenics_shells.fem.assemble** (*args, **kwargs)

Pass-through for dolfin.assemble and fenics_shells.projected_assemble.

If the first argument is an instance of ProjectedFunctionSpace it will call fenics_shells.projected_assemble, otherwise it will pass through to dolfin.assemble.

**fenics_shells.fem.reconstruct_full_space**

\[
\text{reconstruct_full_space}(u_f, u_p, a, L, 
  is_interpolation=False, a_is_symmetric=False, 
  form_compiler_parameters=None)
\]

Given a Function on a projected space \( u_p \in U_P \) and a function in the full space \( u_f \in U_F \) such that \( U_P \subset U_F \), reconstruct the variable \( u_f \) on the full space via direct copy of the matching subfunctions shared between \( U_F \) and \( U_P \) and the local solution of the original problem \( a = L \).

**Parameters**

- \( u_f \) – DOLFIN Function on FullFunctionSpace.
- \( u_p \) – DOLFIN Function on ProjectedFunctionSpace.
- \( a \) – Bilinear form.
- \( L \) – Bilinear form.

**Returns** DOLFIN Function on FullFunctionSpace.

**Return type** \( u_f \)
1.4 fenics_shells.functions package

1.4.1 Submodules

1.4.2 fenics_shells.functions.functionspace module

1.4.3 Module contents

1.5 fenics_shells.kirchhoff_love package

1.5.1 Submodules

1.5.2 fenics_shells.kirchhoff_love.forms module

fenics_shells.kirchhoff_love.forms.theta(w)

Returns the rotations as a function of the transverse displacements according to the Kirchoff-Love kinematics.

..math:: \theta = \nabla w

Parameters

w – Transverse displacement field, typically a UFL scalar or a DOLFIN Function

Returns

the rotations with shape (2,)

1.5.3 Module contents

Overview

This package contains routines for simulating thin plate structures using the Kirchoff-Love theory.

Further reading

1.6 fenics_shells.naghdi package

1.6.1 Submodules

1.6.2 fenics_shells.naghdi.kinematics module

fenics_shells.naghdi.kinematics.G(F)

Returns the stretching tensor (1st non-linear Naghdi strain measure).

\[ G = \frac{1}{2}(F^TF - I) \]

Parameters

F – Deformation gradient.

Returns

UFL expression of stretching tensor.

fenics_shells.naghdi.kinematics.K(F, d)

Returns the curvature tensor (2nd non-linear Naghdi strain measure).

\[ K = \frac{1}{2}(F^T \nabla d + (\nabla d)^T F^T) \]
Parameters

• **F** – Deformation gradient.
• **d** – Director vector.

**Returns** UFL expression of curvature tensor.

```
fenics_shells.naghdi.kinematics.d(theta)
```

Director vector.

\[ d = \begin{bmatrix} \sin(\theta_2)\cos(\theta_1), & -\sin(\theta_1), & \cos(\theta_2)\cos(\theta_1) \end{bmatrix}^T \]

Parameters **vector** (Rotation)

**Returns** UFL expression of director vector.

```
fenics_shells.naghdi.kinematics.g(F, d)
```

Returns the shear strain vector (3rd non-linear Naghdi strain measure).

\[ g = F^T d \]

Parameters

• **F** – Deformation gradient.
• **d** – Director vector.

**Returns** UFL expression of shear strain vector.

### 1.6.3 Module contents

### 1.7 fenics_shells.reissner_mindlin package

#### 1.7.1 Submodules

#### 1.7.2 fenics_shells.reissner_mindlin.forms module

```
fenics_shells.reissner_mindlin.forms.gamma(theta, w)
```

Return shear strain vector calculated from primal variables:

\[ \gamma = \nabla w - \theta \]

```
fenics_shells.reissner_mindlin.forms.inner_e(x, y, restrict_to_one_side=False, quadrature_degree=1)
```

The inner product of the tangential component of a vector field on all of the facets of the mesh (Measure objects dS and ds).

By default, restrict_to_one_side is False. In this case, the function will return an integral that is restricted to both sides (‘+’) and (‘-’) of a shared facet between elements. You should use this in the case that you want to use the ‘projected’ version of DuranLibermanSpace.

If restrict_to_one_side is True, then this will return an integral that is restricted (‘+’) to one side of a shared facet between elements. You should use this in the case that you want to use the multipliers version of DuranLibermanSpace.

Parameters

• **x** – DOLFIN or UFL Function of rank (2,) (vector).
• **y** – DOLFIN or UFL Function of rank $(2,)$ (vector).
• *(Optional*[bool]* (restrict_to_one_side) – Default is False.
• **quadrature_degree**(Optional*[int]*) – Default is 1.

**Returns** UFL Form.

fenics_shells.reissner_mindlin.forms.psi_T(gamma, **kwargs)

Returns transverse shear energy density calculated using:

.. math::
   \psi_T(\gamma, \ldots) = \frac{E \kappa t}{4(1 + \nu)}\gamma^2

**Parameters**

• **gamma** – Shear strain, typically UFL form with shape $(2,)$. 
• ****kwargs** – Isotropic case: E: Young’s modulus, Constant or Expression. nu: Poisson’s ratio, Constant or Expression. t: Thickness, Constant or Expression. kappa: Shear correction factor, Constant or Expression.

**Returns** UFL expression of transverse shear stress vector.

1.7.3 fenics_shells.reissner_mindlin.function_spaces module

fenics_shells.reissner_mindlin.function_spaces.DuranLibermanSpace(mesh)

A helper function which returns a FiniteElement for the simulation of the out-of-plane Reissner-Mindlin problem without shear-locking based on the ideas of Duran and Liberman’s paper:


**Parameters** mesh (dolfin.Mesh) – a mesh of geometric dimension 2.

**Returns** fenics_shells.ProjectedFunctionSpace.

fenics_shells.reissner_mindlin.function_spaces.MITC7Space(mesh, space_type='multipliers')

Warning: Currently not working due to regressions in FFC/FIAT.

A helper function which returns a FunctionSpace for the simulation of the out-of-plane Reissner-Mindlin problem without shear-locking based on the ideas of Brezzi, Bathe and Fortin’s paper:


In the case that space_type is “multipliers”, a dolfin.FunctionSpace will be returned with an additional Lagrange multiplier field to tie the shear strains computer with the primal variables (transverse displacement and rotations) to the independent shear strain variable.

In the case that space_type is “primal”, a dolfin.FunctionSpace will be returned with just the primal (transverse displacement and rotation) variables. Of course, any Reissner-Mindlin problem constructed with this approach will be prone to shear-locking.

**Parameters**

• mesh (dolfin.Mesh) – a mesh of geometric dimension 2.
• space_type (Optional*[str]*) – Can be “primal”, or 
• Default is "multipliers". ("multipliers".) –

1.7. fenics_shells.reissner_mindlin package

11
Returns `dolfin.FunctionSpace`.

1.7.4 Module contents

Overview

This package contains routines for simulating thin through to moderately thick plate structures using the Reissner-Mindlin theory.

Further reading

1.8 `fenics_shells.utils` package

1.8.1 Submodules

1.8.2 `fenics_shells.utils.Probe` module

`fenics_shells.utils.Probe.strip_essential_code(filenames)`

1.8.3 Module contents

1.9 `fenics_shells.von_karman` package

1.9.1 Submodules

1.9.2 `fenics_shells.von_karman.kinematics` module

`fenics_shells.von_karman.kinematics.e(u, theta)`

Return the membrane strain tensor for the Von-Karman plate model.

\[ e(u, \theta) = \text{sym} \nabla u + \frac{\theta \otimes \theta}{2} \]

Parameters

- `u` – In-plane displacement.
- `theta` – Rotations.

Returns UFL form of Von-Karman membrane strain tensor.

1.9.3 Module contents

Overview

This package contains routines for simulating thin plate structures using the von Karman theory.

Further reading
fenics-shells is an open-source library that provides a wide range of thin structural models (beams, plates and shells) expressed in the Unified Form Language (UFL) of the FEniCS Project.
CHAPTER 3

Documented demos

3.1 Where to start

We suggest new users to look at the following two demos for having a first idea of the possible approaches and modelling capabilities for linear plate (MITC discretisation) and nonlinear shells (PRSI discretisation):

3.1.1 Clamped Reissner-Mindlin plate under uniform load

This demo is implemented in the single Python file demo_reissner-mindlin-clamped.py.

This demo program solves the out-of-plane Reissner-Mindlin equations on the unit square with uniform transverse loading with fully clamped boundary conditions. It is assumed the reader understands most of the functionality in the FEniCS Project documented demos.

Specifically, you should know how to:

• Define a MixedElement and a FunctionSpace from it.
• Write variational forms using the Unified Form Language.
• Automatically derive Jacobian and residuals using derivative().
• Apply Dirichlet boundary conditions using DirichletBC and apply().
• Assemble forms using assemble().
• Solve linear systems using LUSolver.
• Output data to XDMF files with XDMFFile.

This demo then illustrates how to:

• Define the Reissner-Mindlin plate equations using UFL.
• Define the Durán-Liberman (MITC) reduction operator using UFL. This procedure eliminates the shear-locking problem.
• Use ProjectedFunctionSpace and assemble() in FEniCS-Shells to statically condensate two problem variables and assemble a linear system of reduced size.
• Reconstruct the variables that were statically condensed using reconstruct_full_space().

First the dolfin and fenics_shells modules are imported. The fenics_shells module overrides some standard methods in DOLFIN, so it should always be import-ed after dolfin:

```python
from dolfin import *
from fenics_shells import *
```

We then create a two-dimensional mesh of the mid-plane of the plate \( \Omega = [0, 1] \times [0, 1] \):

```python
mesh = UnitSquareMesh(32, 32)
```

The Durán-Liberman element for the Reissner-Mindlin plate problem consists of second-order vector-valued element for the rotation field \( \theta \in [CG_2]^2 \) and a first-order scalar valued element for the transverse displacement field \( w \in CG_1 \), see [1]. Two further auxiliary fields are also considered, the reduced shear strain \( \gamma_R \), and a Lagrange multiplier field \( p \) which ties together the shear strain calculated from the primal variables \( \gamma = \nabla w - \theta \) and the reduced shear strain \( \gamma_R \). Both \( p \) and \( \gamma_R \) are are discretised in the space NED_1, the vector-valued Nédélec elements of the first kind. The final element definition is then:

```python
element = MixedElement([VectorElement("Lagrange", triangle, 2),
                       FiniteElement("Lagrange", triangle, 1),
                       FiniteElement("N1curl", triangle, 1),
                       FiniteElement("N1curl", triangle, 1)])
```

We then pass our element through to the ProjectedFunctionSpace constructor. As we will see later in this example, we can project out both the \( p \) and NED_1 fields at assembly time. We specify this by passing the argument num_projected_subspaces=2:
\[ Q = \text{ProjectedFunctionSpace}(\text{mesh}, \text{element}, \text{num_projected_subspaces}=2) \]

From \( Q \) we can then extract the full space \( Q_F \), which consists of all four function fields, collected in the state vector \( q = (\theta, w, \gamma_R, p) \).

\[ Q_F = Q.\text{full_space} \]

In contrast the projected space \( Q \) only holds the two primal problem fields \((\theta, w)\).

Using only the \textit{full} function space object \( Q_F \) we setup our variational problem by defining the Lagrangian of the Reissner-Mindlin plate problem. We begin by creating a \texttt{Function} and splitting it into each individual component function:

\[
\begin{align*}
q_&= \text{Function}(Q_F) \\
\text{theta}_&, \text{w}_&, \text{R}_\gamma&, \text{p}_&= \text{split}(q) \\
q &= \text{TrialFunction}(Q_F) \\
q_t &= \text{TestFunction}(Q_F)
\end{align*}
\]

We assume constant material parameters; Young’s modulus \( E \), Poisson’s ratio \( \nu \), shear-correction factor \( \kappa \), and thickness \( t \):

\[
\begin{align*}
E &= \text{Constant}(10920.0) \\
\nu &= \text{Constant}(0.3) \\
\kappa &= \text{Constant}(5.0/6.0) \\
t &= \text{Constant}(0.001)
\end{align*}
\]

The bending strain tensor \( k \) for the Reissner-Mindlin model can be expressed in terms of the rotation field \( \theta \):

\[ k(\theta) = \frac{1}{2}(\nabla \theta + \nabla \theta^T) \]

which can be expressed in UFL as:

\[ k = \text{sym}(\text{grad}(\text{theta}_\_)) \]

The bending energy density \( \psi_b \) for the Reissner-Mindlin model is a function of the bending strain tensor \( k \):

\[ \psi_b(k) = \frac{1}{2}D \left( (1-\nu) \text{tr}(k^2) + \nu (\text{tr}(k))^2 \right) \quad D = \frac{Et^3}{12(1-\nu^2)} \]

which can be expressed in UFL as:

\[
\begin{align*}
D &= (E*t**3)/(12.0*(1.0 - nu**2)) \\
\text{psi}_b &= 0.5*D*((1.0 - nu)*tr(k*k) + nu*(tr(k))**2)
\end{align*}
\]

Because we are using a mixed variational formulation, we choose to write the shear energy density \( \psi_s \) is a function of the reduced shear strain vector:

\[ \psi_s(\gamma_R) = \frac{E\kappa t}{4(1+\nu)}\gamma_R^2 \]

or in UFL:

\[ \text{psi}_s = ((E*kappa*t)/(4.0*(1.0 + nu))) \text{inner}(R\_gamma\_, R\_gamma\_) \]

Finally, we can write out external work due to the uniform loading in the out-of-plane direction:

\[ W_{ext} = \int_\Omega ft^3 \cdot w \, dx. \]
where \( f = 1 \) and \( dx \) is a measure on the whole domain. The scaling by \( t^3 \) is included to ensure a correct limit solution as \( t \to 0 \).

In UFL this can be expressed as:

```python
f = Constant(1.0)
W_ext = inner(f*t**3, w_)*dx
```

With all of the standard mechanical terms defined, we can turn to defining the numerical Duran-Liberman reduction operator. This operator ‘ties’ our reduced shear strain field to the shear strain calculated in the primal space. A partial explanation of the thinking behind this approach is given in the Appendix.

The shear strain vector \( \gamma \) can be expressed in terms of the rotation and transverse displacement field:

\[
\gamma(\theta, w) := \nabla w - \theta
\]

or in UFL:

```python
gamma = grad(w_) - theta_
```

We require that the shear strain calculated using the displacement unknowns \( \gamma = \nabla w - \theta \) be equal, in a weak sense, to the conforming shear strain field \( \gamma_R \in \text{NED}_1 \) that we used to define the shear energy above. We enforce this constraint using a Lagrange multiplier field \( p \in \text{NED}_1 \). We can write the Lagrangian of this constraint as:

\[
\Pi_R(\gamma, \gamma_R, p) = \int_e (\{\gamma_R - \gamma\} \cdot t) \cdot (p \cdot t) \, ds
\]

where \( e \) are all of edges of the cells in the mesh and \( t \) is the tangent vector on each edge.

Writing this operator out in UFL is quite verbose, so `fenics_shells` includes a special `all edges` inner product function `inner_e()` to help. However, we choose to write the operation out in full here:

```python
dSp = Measure('dS', metadata={'quadrature_degree': 1})
dsp = Measure('ds', metadata={'quadrature_degree': 1})
n = FacetNormal(mesh)
t = as_vector((-n[1], n[0]))

inner_e = lambda x, y: (inner(x, t)*inner(y, t))('+')*dSp + \n                     (inner(x, t)*inner(y, t))('-')*dSp + \n                     (inner(x, t)*inner(y, t))*dsp
Pi_R = inner_e(gamma - R_gamma_, p_)
```

We can now define our Lagrangian for the complete system:

```python
Pi = psi_b*dx + psi_s*dx + Pi_R - W_ext
```

and derive our Jacobian and residual automatically using the standard UFL `derivative()` function:

```python
dPi = derivative(Pi, q_, q_t)
J = derivative(dPi, q_, q)
```

We now assemble our system using the additional projected assembly in `fenics_shells`.

By passing `Q_P` as the first argument to `assemble()`, we state that we want to assemble a Matrix or Vector from the forms on the `ProjectedFunctionSpace Q`, rather than the full `FunctionSpace Q_F`:
\( A, b = \text{assemble}(Q, J, -\text{dPi}) \)

Note that from this point on, we are working with objects on the ProjectedFunctionSpace \( Q \). We now apply homogeneous Dirichlet boundary conditions:

```python
def all_boundary(x, on_boundary):
    return on_boundary

bcs = [DirichletBC(Q, Constant((0.0, 0.0, 0.0)), all_boundary)]
for bc in bcs:
    bc.apply(A, b)
```

and solve the linear system of equations:

```python
q_p_ = Function(Q)
solver = PETScLUSolver("mumps")
solver.solve(A, q_p_.vector(), b)
```

We can now reconstruct the full space solution (i.e. the fields \( \gamma_R \) and \( p \)) using the method `reconstruct_full_space()`:

```
reconstruct_full_space(q_, q_p_, J, -dPi)
```

This step is not necessary if you are only interested in the primal fields \( w \) and \( \theta \).

Finally we output the results to XDMF to the directory `output/`:

```python
save_dir = "output/
theta_h, w_h, R_gamma_h, p_h = q_.split()
fields = {"theta": theta_h, "w": w_h, "R_gamma": R_gamma_h, "p": p_h}
for name, field in fields.items():
    field.rename(name, name)
    field_file = XDMFFile("%s/%s.xdmf" % (save_dir, name))
    field_file.write(field)
```

The resulting `output/*_xdmf` files can be viewed using Paraview.

**Appendix**

For the clamped problem we have the following regularity for our two fields, \( \theta \in [H_0^1(\Omega)]^2 \) and \( w \in [H_0^1(\Omega)]^2 \) where \( H_0^1(\Omega) \) is the usual Sobolev space of functions with square integrable first derivatives that vanish on the boundary. If we then take \( \nabla w \) we have the result \( \nabla w \in H_0(\text{rot}; \Omega) \) which is the Sobolev space of vector-valued functions with square integrable rot whose tangential component \( \nabla w \cdot t \) vanishes on the boundary. Functions \( \nabla w \in H_0(\text{rot}; \Omega) \) are rot free, in that \( \text{rot}(\nabla w) = 0 \).

Let’s look at our expression for the shear strain vector in light of these new results. In the thin-plate limit \( t \to 0 \), we would like to recover our the standard Kirchhoff-Love problem where we do not have transverse shear strains \( \gamma \to 0 \) at all. In a finite element context, where we have discretised fields \( w_h \) and \( \theta_h \) we then would like:

\[
\gamma(\theta_h, w_h) := \nabla w_h - \theta_h = 0 \quad t \to 0 \quad \forall x \in \Omega
\]

If we think about using first-order piecewise linear polynomial finite elements for both fields, then we are requiring that piecewise constant functions \( \nabla w_h \) are equal to piecewise linear functions \( \theta_h \)! This is strong requirement, and is the root of the famous shear-locking problem. The trick of the Durán-Liberman approach is recognising that
by modifying the rotation field at the discrete level by applying a special operator $R_h$ that takes the rotations to the conforming space $\text{NED}_1 \subset H_0(\text{rot}; \Omega)$ for the shear strains that we previously identified:

$$R_h : H^1_0(\Omega) \rightarrow H_0(\text{rot}; \Omega),$$

we can ‘unlock’ the element. With this reduction operator applied as follows:

$$\gamma(\theta_h, w_h) := R_h(\nabla w_h - \theta_h) \quad t \to 0 \quad \forall x \in \Omega$$

our requirement of vanishing shear strains can actually hold. This is the basic mathematical idea behind all MITC approaches, of which the Durán-Liberman approach is a subclass.

**Unit testing**

```python
def test_close():
    import numpy as np
    assert(np.isclose(w_h((0.5, 0.5)), 1.285E-6, atol=1E-3, rtol=1E-3))
```

**References**


**3.1.2 Clamped semi-cylindrical Naghdi shell under point load**

This demo is implemented in the single Python file `demo_nonlinear-naghdi-cylindrical.py`.

This demo program solves the nonlinear Naghdi shell equations for a semi-cylindrical shell loaded by a point force. This problem is a standard reference for testing shell finite element formulations, see [1]. The numerical locking issue is cured using enriched finite element including cubic bubble shape functions and Partial Selective Reduced Integration [2].

To follow this demo you should know how to:

- Define a `MixedElement` and a `FunctionSpace` from it.
- Write variational forms using the Unified Form Language.
- Automatically derive Jacobian and residuals using `derivative()`.
- Apply Dirichlet boundary conditions using `DirichletBC` and `apply()`.
- Solve non-linear problems using `NonlinearProblem`.
- Output data to XDMF files with `XDMFFile`.

This demo then illustrates how to:

- Define and solve a nonlinear Naghdi shell problem with a curved stress-free configuration given as analytical expression in terms of two curvilinear coordinates.
- Use the PSRI approach to simultaneously cure shear- and membrane-locking issues.

We start with importing the required modules, setting `matplotlib` as plotting backend, and generically set the integration order to 4 to avoid the automatic setting of FEniCS which would lead to unreasonably high integration orders for complex forms.
Fig. 3.2: Shell deformed configuration.
We consider a semi-cylindrical shell of radius \( \rho \) and axis length \( L \). The shell is made of a linear elastic isotropic homogeneous material with Young modulus \( E \) and Poisson ratio \( \nu \). The (uniform) shell thickness is denoted by \( t \). The Lamé moduli \( \lambda, \mu \) are introduced to write later the 2D constitutive equation in plane-stress:

\[
E_{\text{eff}} = \frac{E}{2(1+\nu)}
\]

\[
\mu_{\text{eff}} = \frac{\mu}{1-2\nu}
\]

\[
t = \text{Constant}(0.03)
\]

The midplane of the initial (stress-free) configuration \( \Phi_0 \) of the shell is given in the form of an analytical expression

\[
\phi_0 : x \in \omega \subset \mathbb{R}^2 \rightarrow \phi_0(x) \in \Phi_0 \subset \mathbb{R}^3
\]

in terms of the curvilinear coordinates \( x \). In the specific case we adopt the cylindrical coordinates \( x_0 \) and \( x_1 \)
representing the angular and axial coordinates, respectively. Hence we mesh the two-dimensional domain \( \omega \equiv [0, L] \times [-\pi/2, \pi/2] \).

We provide the analytical expression of the initial shape as an Expression that we represent on a suitable FunctionSpace (here \( P_2 \), but other choices are possible):

\[
\text{initial\_shape} = \text{Expression}(('r*sin(x[0])','x[1]','r*cos(x[0])'), r=rho, degree = 4)
\]

\[
V\_phi = \text{FunctionSpace}(\text{mesh}, \text{VectorElement}("P", \text{triangle}, \text{degree} = 2, \text{dim} = 3))
\]

\[
\phi_0 = \text{project}(\text{initial\_shape}, V\_phi)
\]

Given the midplane, we define the corresponding unit normal as below and project on a suitable function space (here \( P_1 \) but other choices are possible):

\[
\text{normal}(y) = \frac{\text{cross}(y.dx(0), y.dx(1))}{\sqrt{\text{inner}(n,n)}}
\]

\[
V\_normal = \text{FunctionSpace}(\text{mesh}, \text{VectorElement}("P", \text{triangle}, \text{degree} = 1, \text{dim} = 3))
\]

\[
n_0 = \text{project}(\text{normal}(\phi_0), V\_normal)
\]
Fig. 3.3: Discretisation of the parametric domain.
The kinematics of the Nadghi shell model is defined by the following vector fields:

- \( \phi \): the position of the midplane, or the displacement from the reference configuration \( u = \phi - \phi_0 \):

- \( d \): the director, a unit vector giving the orientation of the microstructure

We parametrize the director field by two angles, which correspond to spherical coordinates, so as to explicitly resolve the unit norm constraint (see [3]):

```python
def director(beta):
    return as_vector([sin(beta[1])*cos(beta[0]), -sin(beta[0]),
                      cos(beta[1])*cos(beta[0])])
```

We assume that in the initial configuration the director coincides with the normal. Hence, we can define the angles \( \beta \): for the initial configuration as follows:

```python
beta0_expression = Expression(["atan2(-n[1], sqrt(pow(n[0],2) + pow(n[2],2)))",
                              "atan2(n[0],n[2])"], n = n0, degree=4)
V_beta = FunctionSpace(mesh, VectorElement("P", triangle, degree = 2, dim = 2))
beta0 = project(beta0_expression, V_beta)
```

The director in the initial configuration is then written as

```python
d0 = director(beta0)
```

We can visualize the shell shape and its normal with this utility function:

```python
def plot_shell(y, n=None):
    y_0, y_1, y_2 = y.split(deepcopy=True)
    fig = plt.figure()
    ax = fig.gca(projection='3d')
    ax.plot_trisurf(y_0.compute_vertex_values(),
                    y_1.compute_vertex_values(),
                    y_2.compute_vertex_values(),
                    triangles=y.function_space().mesh().cells(),
                    linewidth=1, antialiased=True, shade=False)
    if n:
        n_0, n_1, n_2 = n.split(deepcopy=True)
        ax.quiver(y_0.compute_vertex_values(),
                  y_1.compute_vertex_values(),
                  y_2.compute_vertex_values(),
                  n_0.compute_vertex_values(),
                  n_1.compute_vertex_values(),
                  n_2.compute_vertex_values(),
                  length=.2, color="r")
    ax.view_init(elev=20, azim=80)
    plt.xlabel(r"$x_0$")
    plt.ylabel(r"$x_1$")
    plt.xticks([-1,0,1])
    plt.yticks([0,pi/2])
    return ax
plot_shell(phi0, project(d0, V_normal))
plt.savefig("output/initial_configuration.png")
```

In our 5-parameter Naghdi shell model the configuration of the shell is assigned by

- the 3-component vector field \( u \): representing the displacement with respect to the initial configuration \( \phi_0 \):
Fig. 3.4: Shell initial shape and normal.
- the 2-component vector field $\beta$: representing the angle variation of the director $d$: with respect to the initial configuration

Following [1], we use a $[P_2 + B_3]$ element for $u$ and a $[CG_2]^2$ element for $\beta$, and collect them in the state vector $q = (u, \beta)$:

```python
P2 = FiniteElement("Lagrange", triangle, degree = 2)
bubble = FiniteElement("B", triangle, degree = 3)
enriched = P2 + bubble
element = MixedElement([VectorElement(enriched, dim=3), VectorElement(P2, dim=2)])
Q = FunctionSpace(mesh, element)
```

Then, we define `Function`, `TrialFunction` and `TestFunction` objects to express the variational forms and we split them into each individual component function:

```python
q_, q, q_t = Function(Q), TrialFunction(Q), TestFunction(Q)
u_, beta_ = split(q_)
```

The gradient of the transformation and the director in the current configuration are given by:

```python
F = grad(u_) + grad(phi0)
d = director(beta_ + beta0)
```

With the following definition of the natural metric and curvature

```python
a0 = grad(phi0).T*grad(phi0)
b0 = -0.5*(grad(phi0).T*grad(d0) + grad(d0).T*grad(phi0))
```

The membrane, bending, and shear strain measures of the Naghdi model are defined by:

```python
e = lambda F: 0.5*(F.T*F - a0)
k = lambda F, d: -0.5*(F.T*grad(d) + grad(d).T*F) - b0
gamma = lambda F, d: F.T*d - grad(phi0).T*d0
```

Using curvilinear coordinates, and denoting by $a_{0\alpha\beta}^{\text{contra}}$ the contravariant components of the metric tensor $a_0^{\alpha\beta}$ (in the initial curved configuration) the constitutive equation is written in terms of the matrix $A$ below, representing the contravariant components of the constitutive tensor for isotropic elasticity in plane stress (see e.g. [4]). We use the index notation offered by UFL to express operations between tensors:

```python
A_ = as_tensor((((2.0*lmbda*mu)/(lmbda + 2.0*mu))*a0_contra[i,j]*a0_contra[l,m]
    + 1.0*mu*(a0_contra[i,l]*a0_contra[j,m] + a0_contra[i,m]*a0_contra[j,l]))
    ,[i,j,l,m])
```

The normal stress $N$, bending moment $M$, and shear stress $T$ tensors are (they are purely Lagrangian stress measures, similar to the so called 2nd Piola stress tensor in 3D elasticity):

```python
N = as_tensor(t*A_[i,j,l,m]*e(F)[l,m], [i,j])
M = as_tensor((t*3/12.0)*A_[i,j,l,m]*k(F,d)[l,m], [i,j])
T = as_tensor(t*mu*a0_contra[i,j]*gamma(F,d)[j], [i])
```
Hence, the contributions to the elastic energy density due to membrane, $\psi_m$, bending, $\psi_b$, and shear, $\psi_s$ are (they are per unit surface in the initial configuration):

\[
\begin{align*}
\psi_m &= 0.5 \times \text{inner}(N, e(F)) \\
\psi_b &= 0.5 \times \text{inner}(M, k(F,d)) \\
\psi_s &= 0.5 \times \text{inner}(T, \gamma(F,d))
\end{align*}
\]

Shear and membrane locking is treated using the partial reduced selective integration proposed in Arnold and Brezzi [2]. In this approach shear and membrane energy are split as a sum of two contributions weighted by a factor $\alpha$. One of the two contributions is integrated with a reduced integration. While [1] suggests a 1-point reduced integration, we observed that this leads to spurious modes in the present case. We use then $2 \times 2$-points Gauss integration for a portion $1 - \alpha$ of the energy, whilst the rest is integrated with a $4 \times 4$ scheme. We further refine the approach of [1] by adopting an optimized weighting factor $\alpha = (t/h)^2$, where $h$ is the mesh size.

\[
\begin{align*}
\text{dx}_h &= \text{dx} \left( \text{metadata}='\text{quadrature\_degree': 2} \right) \\
h &= \text{CellDiameter} \left( \text{mesh} \right) \\
\alpha &= \text{project} \left( \text{t}^2/h^2, \text{FunctionSpace} \left( \text{mesh}, 'DG', 0 \right) \right)
\end{align*}
\]

Pi_PSRI = $\psi_b \sqrt{j0} \times dx + \alpha \times \psi_m \sqrt{j0} \times dx + \alpha \times \psi_s \sqrt{j0} \times dx + (1.0 - \alpha) \times \psi_s \sqrt{j0} \times dx_h$

Hence the total elastic energy and its first and second derivatives are

\[
\begin{align*}
\text{Pi} &= \text{Pi}_\text{PSRI} \\
\text{dPi} &= \text{derivative} \left( \text{Pi}, q_{-}, q_t \right) \\
\text{J} &= \text{derivative} \left( \text{dPi}, q_{-}, q \right)
\end{align*}
\]

The boundary conditions prescribe a full clamping on the top boundary, while on the left and the right side the normal component of the rotation and the transverse displacement are blocked:

\[
\begin{align*}
\text{up\_boundary} &= \text{lambda} \ x, \ \text{on\_boundary}: x[1] \leq 1.e-4 \ \text{and} \ \text{on\_boundary} \\
\text{letright\_boundary} &= \text{lambda} \ x, \ \text{on\_boundary}: \text{near} \left( \text{abs} \left( x[0] \right), \pi/2., \ 1.e-6 \right) \ \text{and} \ \text{on\_boundary}
\end{align*}
\]

\[
\begin{align*}
\text{bc\_clamped} &= \text{DirichletBC} \left( Q, \ \text{project} \left( q_{-}, Q, \ \text{up\_boundary} \right) \right) \\
\text{bc\_u} &= \text{DirichletBC} \left( Q.\text{sub}(0).\text{sub}(2), \ \text{project} \left( \text{Constant}(0.), Q.\text{sub}(0).\text{sub}(2).\text{collapse()} \right), \ \text{letright\_boundary} \right) \\
\text{bc\_beta} &= \text{DirichletBC} \left( Q.\text{sub}(1).\text{sub}(1), \ \text{project} \left( q_4, Q.\text{sub}(1).\text{sub}(0).\text{collapse()} \right), \ \text{letright\_boundary} \right) \\
\text{bcs} &= [\text{bc\_clamped}, \text{bc\_u}, \text{bc\_beta}]
\end{align*}
\]

The loading is exerted by a point force applied at the midpoint of the bottom boundary. This is implemented using the PointSource in FEniCS and defining a custom NonlinearProblem:

```python
class NonlinearProblemPointSource(NonlinearProblem):
    def __init__(self, L, a, bcs):
        NonlinearProblem.__init__(self)
        self.L = L
        self.a = a
        self.bcs = bcs
        self.P = 0.0

    def F(self, b, x):
        assemble(self.L, tensor=b)
        point_source = PointSource(self.bcs[0].function_space().sub(0).sub(2).collapse())
        Point(0.0, L), self.P
        point_source.apply(b)
```

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for bc in self.bcs:
    bc.apply(b, x)

def J(self, A, x):
    assemble(self.a, tensor=A)
    for bc in self.bcs:
        bc.apply(A, x)

problem = NonlinearProblemPointSource(dPi, J, bcs)

We use a standard Newton solver and setup the files for the writing the results to disk:

```python
solver = NewtonSolver()
output_dir = "output/"
file_phi = File(output_dir + "configuration.pvd")
file_energy = File(output_dir + "energy.pvd")
```

Finally, we can solve the quasi-static problem, incrementally increasing the loading from 0 to 2000 N:

```python
P_values = np.linspace(0.0, 2000.0, 40)
displacement = 0.*P_values
q_.assign(project(Constant((0,0,0,0,0)), Q))
for (i, P) in enumerate(P_values):
    problem.P = P
    (niter, cond) = solver.solve(problem, q_.vector())

    phi = project(u_ + phi0, V_phi)
    displacement[i] = phi(0.0, L)[2] - phi0(0.0, L)[2]

    phi.rename("phi", "phi")
    file_phi << (phi, P)
    print("Increment %d of %s. Converged in %2d iterations. P: %.2f, Displ: %.2f" %
          (i, P_values.size,niter,P, displacement[i]))

    en_function = project(psi_m + psi_b + psi_s, FunctionSpace(mesh, 'Lagrange', 1))
    en_function.rename("Elastic Energy", "Elastic Energy")
    file_energy << (en_function,P)
```

We can plot the final configuration of the shell:

```python
plot_shell(phi)
plt.savefig("output/finalconfiguration.png")
```

The results for the transverse displacement at the point of application of the force are validated against a standard reference from the literature, obtained using Abaqus S4R element and a structured mesh of 40 × 40 elements, see [1]:

```python
plt.figure()
reference_Sze = np.array([ 1.e-2*np.array([0., 5.421, 16.1, 22.195, 27.657, 32.7, 37.582, 42.633,
        48.537, 56.355, 66.410, 79.810, 94.669, 113.704, 124.751, 132.653,
        138.920, 144.185, 148.770, 152.863, 156.584, 160.015, 163.211,
        166.973, 171.505]),
        2000.*np.array([0., .05, .1, .125, .15, .175, .2, .225, .25, .275, .3,
        .325, .35, .4, .45, .5, .55, .6, .65, .7, .75, .8, .85, .9, .95, 1.])])
plt.plot(-np.array(displacement), P_values, label='fenics-shell %s divisions (AB)' %ndiv)
```
Fig. 3.5: Shell deformed shape.
plt.plot(*reference_Sze, "or", label='Sze (Abaqus S4R)')
plt.xlabel("Displacement (mm)")
plt.ylabel("Load (N)")
plt.legend()
plt.grid()
plt.savefig("output/comparisons.png")

Fig. 3.6: Comparison with reference solution.

References


3.2 Other demos

Linear Reissner-Mindlin plate problems using the Durán-Liberman reduction operator (MITC) and MITC7 to cure shear-locking:

3.2.1 Simply supported Reissner-Mindlin plate

This demo is implemented in the single Python file demo_reissner-mindlin-simply-supported.py. This demo program solves the out-of-plane Reissner-Mindlin equations on the unit square with uniform transverse loading with simply supported boundary conditions.

The first part of the demo is similar to the demo Clamped Reissner-Mindlin plate under uniform load.

```python
from dolfin import *
from fenics_shells import *

mesh = UnitSquareMesh(64, 64)

element = MixedElement([VectorElement("Lagrange", triangle, 2),
                        FiniteElement("Lagrange", triangle, 1),
                        FiniteElement("N1curl", triangle, 1),
                        FiniteElement("N1curl", triangle, 1)])

U = ProjectedFunctionSpace(mesh, element, num_projected_subspaces=2)
U_F = U.full_space
u_ = Function(U_F)
theta_, w_, R_gamma_, p_ = split(u_)
u = TrialFunction(U_F)
u_t = TestFunction(U_F)

E = Constant(10920.0)
nu = Constant(0.3)
kappa = Constant(5.0/6.0)
t = Constant(0.0001)

k = sym(grad(theta_))

D = (E*t**3)/(24.0*(1.0 - nu**2))
psi_M = D*((1.0 - nu)*tr(k*k) + nu*(tr(k))**2)
psi_T = ((E*kappa*t)/(4.0*(1.0 + nu)))*inner(R_gamma_, R_gamma_)
f = Constant(1.0)
W_ext = inner(f*t**3, w_)*dx

L_R = inner_e(gamma - R_gamma_, p_)
L = psi_M*dx + psi_T*dx + L_R - W_ext
F = derivative(L, u_, u_t)
J = derivative(F, u_, u)
```

We instead use the fenics_shells provided inner_e() function:

```python
L_R = inner_e(gamma - R_gamma_, p_)
L = psi_M*dx + psi_T*dx + L_R - W_ext
F = derivative(L, u_, u_t)
J = derivative(F, u_, u)
```
and apply simply-supported boundary conditions:

```python
def all_boundary(x, on_boundary):
    return on_boundary

def left(x, on_boundary):
    return on_boundary and near(x[0], 0.0)

def right(x, on_boundary):
    return on_boundary and near(x[0], 1.0)

def bottom(x, on_boundary):
    return on_boundary and near(x[1], 0.0)

def top(x, on_boundary):
    return on_boundary and near(x[1], 1.0)

# Simply supported boundary conditions.
bcs = [DirichletBC(U.sub(1), Constant(0.0), all_boundary),
      DirichletBC(U.sub(0).sub(0), Constant(0.0), top),
      DirichletBC(U.sub(0).sub(0), Constant(0.0), bottom),
      DirichletBC(U.sub(0).sub(1), Constant(0.0), left),
      DirichletBC(U.sub(0).sub(1), Constant(0.0), right)]

for bc in bcs:
    bc.apply(A, b)
```

and solve the linear system of equations before writing out the results to files in `output/`:

```python
u_p_ = Function(U)
solver = PETScLUSolver("mumps")
solver.solve(A, u_p_.vector(), b)
reconstruct_full_space(u_, u_p_, J, -F)
```

```python
save_dir = "output/
theta, w, R_gamma, p = u_.split()
fields = {
    "theta": theta,
    "w": w,
    "R_gamma": R_gamma,
    "p": p
}
for name, field in fields.items():
    field.rename(name, name)
    field_file = XDMFFile("%s/%s.xdmf" % (save_dir, name))
    field_file.write(field)
```

We check the result against an analytical solution calculated using a series expansion:

```python
from fenics_shells.analytical.simply_supported import Displacement

w_e = Displacement(degree=3)
w_e.t = t.values()
w_e.E = E.values()
w_e.p = f.values() * t.values()**3
w_e.nu = nu.values()

print("Numerical out-of-plane displacement at centre: %.4e" % w((0.5, 0.5)))
print("Analytical out-of-plane displacement at centre: %.4e" % w_e((0.5, 0.5)))
```
Unit testing

```python
def test_close():
    import numpy as np
    assert np.isclose(w((0.5, 0.5)), w_e((0.5, 0.5)), atol=1E-3, rtol=1E-3)
```

3.2.2 Clamped Reissner-Mindlin plate with MITC7

This demo is implemented in a single Python file `demo_reissner-mindlin-mitc7.py`.

This demo program solves the out-of-plane Reissner-Mindlin equations on the unit square with uniform transverse loading with fully clamped boundary conditions. The MITC7 reduction operator is used, instead of the Durán Liberman one as in the `Clamped Reissner-Mindlin plate under uniform load` demo.

We express the MITC7 projection operator in the Unified Form Language. Lagrange multipliers are required on the edge and in the interior of each element to tie together the shear strain calculated from the primal variables and the conforming shear strain field.

Unlike the other Reissner-Mindlin documented demos, e.g. `Clamped Reissner-Mindlin plate under uniform load`, where the FEniCS-Shells `assemble()` function is used to eliminate all of the additional degrees of freedom at assembly time, we keep the full problem with all of the auxiliary variables here.

We begin as usual by importing the required modules:

```python
from dolfin import *
from ufl import EnrichedElement, RestrictedElement
from fenics_shells import *
```

and creating a mesh:

```python
mesh = UnitSquareMesh(32, 32)
```

The MITC7 element for the Reissner-Mindlin plate problem consists of:

- a second-order scalar-valued element for the transverse displacement field \( w \in \text{CG}_2 \),
- second-order bubble-enriched vector-valued element for the rotation field \( \theta \in [\text{CG}_2]^2 \),
- the reduced shear strain \( \gamma_R \) is discretised in the space \( \text{NED}_2 \), the second-order vector-valued Nédélec element of the first kind,
- and two Lagrange multiplier fields \( p \) and \( r \) which tie together the shear strain calculated from the primal variables \( \gamma = \nabla w - \theta \) and the reduced shear strain \( \gamma_R \). \( p \) and \( r \) are discretised in the space \( \text{NED}_2 \) restricted to the element edges and the element interiors, respectively.

The final element definition is:

```python
element = MixedElement([FiniteElement("Lagrange", triangle, 2),
    VectorElement(EnrichedElement(FiniteElement("Lagrange", triangle, 2) + FiniteElement("Bubble", triangle, 3))),
    FiniteElement("N1curl", triangle, 2),
    RestrictedElement(FiniteElement("N1curl", triangle, 2), "edge"),
    VectorElement("DG", triangle, 0))
```

The definition of the bending and shear energies and external work are standard and identical to those in `Clamped Reissner-Mindlin plate under uniform load`:
U = FunctionSpace(mesh, element)

w_, theta_, R_gamma_, p_, r_ = split(u_)
u = TrialFunction(U)
u_t = TestFunction(U)

E = Constant(10920.0)
nu = Constant(0.3)
kappa = Constant(5.0/6.0)
t = Constant(0.001)

k = sym(grad(theta_))
D = (E*t**3)/(24.0*(1.0 - nu**2))
psi_M = D*((1.0 - nu)*tr(k*k) + nu*(tr(k)*tr(k))**2)
psi_T = ((E*kappa*t)/(4.0*(1.0 + nu)))*inner(R_gamma_, R_gamma_)

W_ext = inner(f*t**3, w_)*dx

We require that the shear strain calculated using the displacement unknowns \( \gamma = \nabla w - \theta \) be equal, in a weak sense, to the conforming shear strain field \( \gamma_R \in \text{NED}_2 \) that we used to define the shear energy above. We enforce this constraint using two Lagrange multiplier fields \( p \in \text{NED}_2 \) restricted to the edges and \( r \in \text{NED}_2 \) restricted to the interior of the element. We can write the Lagrangian of this constraint as:

\[
L_R(\gamma, \gamma_R, p, r) = \int_{e} \{(\gamma_R - \gamma) \cdot t \} \cdot (p \cdot t) \, ds + \int_{T} \{\gamma_R - \gamma\} \cdot r \, dx
\]

where \( T \) are all cells in the mesh, \( e \) are all of edges of the cells in the mesh and \( t \) is the tangent vector on each edge.

This operator can be written in UFL as:

n = FacetNormal(mesh)
t = as_vector((-n[1], n[0]))

gamma = grad(w_) - theta_

inner_e = lambda x, y: (inner(x, t)*inner(y, t))('+')*dS + (inner(x, t)*inner(y, t))*ds
inner_T = lambda x, y: inner(x, y)*dx

L_R = inner_e(gamma - R_gamma_, p_) + inner_T(gamma - R_gamma_, r_)

We set homogeneous Dirichlet conditions for the reduced shear strain, edge Lagrange multipliers, transverse displacement and rotations:

```python
def all_boundary(x, on_boundary):
    return on_boundary

bcs = [DirichletBC(U.sub(0), Constant(0.0), all_boundary),
       DirichletBC(U.sub(1), project(Constant((0.0, 0.0)), U.sub(1).collapse()), all_boundary),
       DirichletBC(U.sub(2), Constant((0.0, 0.0)), all_boundary),
       DirichletBC(U.sub(3), Constant((0.0, 0.0)), all_boundary)]
```

Before assembling in the normal way:
\[ L = \psi_M dx + \psi_T dx + L_R - W_{\text{ext}} \]

\[ F = \text{derivative}(L, u, u_t) \]

\[ J = \text{derivative}(F, u, u) \]

\[ A, b = \text{assemble}_\text{system}(J, -F, \text{bcs=bc}) \]

\[ \text{solver} = \text{PETScLU}\text{Solver("mumps")} \]

\[ \text{solver.solve}(A, u_.\text{vector}(), b) \]

\[ w_, \theta_, r_\gamma_, p_, r_ = u_.\text{split}() \]

Finally we output the results to XDMF to the directory `output/`:

```python
save_dir = "output"
fields = {"theta": theta_, "w": w_}
for name, field in fields.items():
    field.rename(name, name)
    field_file = XDMFFile("%s/%s.xdmf"%(save_dir,name))
    field_file.write(field)
```

The resulting `output/*.xdmf` files can be viewed using Paraview.

### Unit testing

```python
def test_close():
    import numpy as np
    assert(np.isclose(w_((0.5, 0.5)), 1.265E-6, atol=1E-3, rtol=1E-3))
```

Fourth-order Kirchhoff-Love plate problem with a Continuous-Discontinuous Galerkin method:

### 3.2.3 Clamped Kirchhoff-Love plate

This demo is implemented in a single Python file `demo_kirchhoff-love-clamped.py`.

This demo program solves the out-of-plane Kirchhoff-Love equations on the unit square with uniform transverse loading and fully clamped boundary conditions.

We use the Continuous/Discontinuous Galerkin (CDG) formulation to allow the use of $H^1$-conforming elements for this fourth-order, or $H^2$-type problem. This demo is very similar to the Biharmonic equation demo in the main DOLFIN repository, and as such we recommend reading that page first. The main differences are:

- we express the stabilisation term in terms of a Lagrangian functional rather than as a bilinear form,
- the Lagrangian function in turn is expressed in terms of the bending moment and rotations rather than the primal field variable,
- and we show how to place Dirichlet boundary conditions on the first-derivative of the solution (the rotations) using a weak approach.

We begin as usual by importing the required modules:

```python
from dolfin import *
from fenics_shells import *
```

and creating a mesh:
FEniCS-Shells, Release 2018.1.0

```python
mesh = UnitSquareMesh(64, 64)

We use a second-order scalar-valued element for the transverse displacement field \( w \in CG_2 \):

```python
element_W = FiniteElement("Lagrange", triangle, 2)
W = FunctionSpace(mesh, element_W)
```

and then define `Function`, `TrialFunction` and `TestFunction` objects to express the variational form:

```python
w_ = Function(W)
w = TrialFunction(W)
w_t = TestFunction(W)
```

We take constant material properties throughout the domain:

```python
E = Constant(10920.0)
nu = Constant(0.3)
t = Constant(1.0)
```

The Kirchhoff-Love model, unlike the Reissner-Mindlin model, is a `rotation-free` model: the rotations \( \theta \) do not appear explicitly as degrees of freedom. Instead, the rotations of the Kirchhoff-Love model are calculated from the transverse displacement field as:

\[
\theta = \nabla w
\]

which can be expressed in UFL as:

```python
theta = grad(w_)
```

The bending tensor can then be calculated from the derived rotation field in exactly the same way as for the Reissner-Mindlin model:

\[
k = \frac{1}{2}(\nabla \theta + (\nabla \theta)^T)
\]

or in UFL:

```python
k = variable(sym(grad(theta)))
```

The function `variable()` annotation is important and will allow us to take differentiate with respect to the bending tensor \( k \) to derive the bending moment tensor, as we will see below.

Again, identically to the Reissner-Mindlin model we can calculate the bending energy density as:

\[
D = \frac{(E*t**3)}{(24.0*(1.0 - nu**2))}
\]

```python
psi_M = D*((1.0 - nu)*tr(k*k) + nu*(tr(k))**2)
```

For the definition of the CDG stabilisation terms and the (weak) enforcement of the Dirichlet boundary conditions on the rotation field, we need to explicitly derive the moment tensor \( M \). Following standard arguments in elasticity, a stress measure (here, the moment tensor) can be derived from the bending energy density by taking its derivative with respect to the strain measure (here, the bending tensor):

\[
M = \frac{\partial \psi_M}{\partial k}
\]

or in UFL:
\[ M = \text{diff}(\psi_M, k) \]

We now move onto the CDG stabilisation terms.

Consider a triangulation \( \mathcal{T} \) of the domain \( \omega \) with the set of interior edges is denoted \( \mathcal{E}_h^{\text{int}} \). Normals to the edges of each facet are denoted \( n \). Functions evaluated on opposite sides of a facet are indicated by the subscripts \( + \) and \( - \).

The Lagrangian formulation of the CDG stabilisation term is then:

\[
L_{\text{CDG}}(w) = \sum_{E \in \mathcal{E}_h^{\text{int}}} \int_{E} \left[ \left[ \theta \right] \right] \cdot \left\langle M \cdot (n \otimes n) \right\rangle + \frac{1}{2} \frac{\alpha}{(h_E)} \left\langle \theta \cdot n \right\rangle \cdot \left\langle \theta \cdot n \right\rangle \, ds
\]

Furthermore, \( \langle u \rangle = \frac{1}{2}(u_+ + u_-) \) operator, \([u] = u_+ \cdot n_+ + u_- \cdot n_-\), \( \alpha \geq 0 \) is a penalty parameter and \( h_E \) is a measure of the cell size. We choose the penalty parameter to be on the order of the norm of the bending stiffness matrix \( \frac{E t^3}{12} \).

This can be written in UFL as:

```python
alpha = E*t**3
h = CellDiameter(mesh)
h_avg = (h('+') + h('-'))/2.0
n = FacetNormal(mesh)
M_n = inner(M, outer(n, n))
L_CDG = -inner(jump(theta, n), avg(M_n))*dS + 
       (1.0/2.0)*(alpha('+')/h_avg)*inner(jump(theta, n), jump(theta, n))*dS
```

We now define our Dirichlet boundary conditions on the transverse displacement field:

```python
class AllBoundary(SubDomain):
    def inside(self, x, on_boundary):
        return on_boundary

# Boundary conditions on displacement
all_boundary = AllBoundary()
bc_w = [DirichletBC(W, Constant(0.0), all_boundary)]
```

Because the rotation field \( \theta \) does not enter our weak formulation directly, we must weakly enforce the Dirichlet boundary condition on the derivatives of the transverse displacement \( \nabla w \).

We begin by marking the exterior facets of the mesh where we want to apply boundary conditions on the rotation:

```python
facet_function = MeshFunction("size_t", mesh, mesh.geometry().dim() - 1)
facet_function.set_all(0)
all_boundary.mark(facet_function, 1)
```

and then define an exterior facet Measure object from that subdomain data:

```python
ds = Measure("ds") (subdomain_data=facet_function)
```

In this example, we would like \( \theta_d = 0 \) everywhere on the boundary:

```python
theta_d = Constant((0.0, 0.0))
```

The definition of the exterior facets and Dirichlet rotation field were trivial in this demo, but you could extend this code straightforwardly to non-homogeneous Dirichlet conditions.
The weak boundary condition enforcement term can be written:

\[ L_{BC}(w) = \sum_{E \in \mathcal{E}^D} \int_E -\theta_e \cdot (M \cdot (n \otimes n)) + \frac{1}{2} \frac{\alpha}{h} \theta_e \cdot n \cdot (\theta_e \cdot n) \, ds \]

where \( \theta_e = \theta - \theta_d \) is the effective rotation field, and \( \mathcal{E}^D \) is the set of all exterior facets of the triangulation \( \mathcal{T} \) where we would like to apply Dirichlet boundary conditions, or in UFL:

```python
theta_effective = theta - theta_d
L_BC = -inner(inner(theta_effective, n), M_n)*ds(1) + \
    (1.0/2.0)*(alpha/h)*inner(inner(theta_effective, n), inner(theta_effective, ...
    \n)*ds(1)
```

The remainder of the demo is as usual:

```python
f = Constant(1.0)
W_ext = f*w_*dx
L = psi_M*dx - W_ext + L_CDG + L_BC
F = derivative(L, w_, w_t)
J = derivative(F, w_, w)
A, b = assemble_system(J, -F, bcs=bcs_w)
solver = PETScLUSolver("mumps")
solver.solve(A, w_.vector(), b)
XDMFFile("output/w.xdmf").write(w_)
```

### Unit testing

```python
def test_close():
    import numpy as np
    assert(np.isclose(w_((0.5, 0.5)), 1.265E-6, atol=1E-3, rtol=1E-3))
```

Von-Kármán plate problem solved with MITC and PRSI approaches to cure locking:

#### 3.2.4 Buckling of a heated von-Kármán plate

This demo is implemented in the single Python file `demo_von-karman-mansfield.py`.  
This demo program solves the von-Kármán equations on a circular plate with a lenticular cross section free on the boundary. The plate is heated, causing it to bifurcate. Bifurcation occurs with a striking shape transition: before the critical threshold the plate assumes a cup-shaped configuration (left); above it tends to a cylindrical shape (right). An analytical solution has been found by Mansfield, see [1].  

The standard von-Kármán theory gives rise to a fourth-order PDE which requires the transverse displacement field \( w \) to be sought in the space \( H^2(\Omega) \). We relax this requirement in the same manner as the Kirchoff-Love plate theory can be relaxed to the Reissner-Mindlin theory, resulting in seeking a transverse displacement field \( w \) in \( H^1(\Omega) \) and a rotation field \( \theta \) in \( [H^2(\Omega)]^2 \). To alleviate the resulting shear- and membrane-locking issues we use Partial Selective Reduced Integration (PSRI), see [2].  

To follow this demo you should know how to:

- Define a `MixedElement` and a `FunctionSpace` from it.
- Write variational forms using the Unified Form Language.
Fig. 3.7: Pre-critical and post-critical plate configuration.

- Automatically derive Jacobian and residuals using `derivative()`.
- Apply Dirichlet boundary conditions using `DirichletBC` and `apply()`.
- Assemble forms using `assemble()`.

This demo then illustrates how to:

- Define the Reissner-Mindlin-von-Kármán plate equations using UFL.
- Use the PSRI approach to simultaneously cure shear- and membrane-locking issues.

We start with importing the required modules, setting `matplotlib` as plotting backend, and generically set the integration order to 4 to avoid the automatic setting of FEniCS which would lead to unreasonably high integration orders for complex forms.

```python
from dolfin import *
from fenics_shells import *
import mshr
import numpy as np
import matplotlib.pyplot as plt

parameters["form_compiler"]["quadrature_degree"] = 4
```

The mid-plane of the plate is a circular domain with radius $a = 1$. We generate a mesh of the domain using the package `mshr`:

```python
a = 1.
ndiv = 8
domain_area = np.pi*a**2
centre = Point(0.,0.)
geom = mshr.Circle(centre, a)
mesh = mshr.generate_mesh(geom, ndiv)
```

The lenticular thinning of the plate can be modelled directly through the thickness parameter in the plate model:

```python
t = 1E-2
ts = interpolate(Expression('t*(1.0 - (x[0]*x[0] + x[1]*x[1])/(a*a))', t=t, a=a,˓→degree=2), FunctionSpace(mesh, 'CG', 2))
```
We assume the plate is isotropic with constant material parameters, Young’s modulus $E$, Poisson’s ratio $\nu$; shear correction factor $\kappa$:

\[E = \text{Constant}(1.0)\]
\[\nu = \text{Constant}(0.3)\]
\[\kappa = \text{Constant}(5.0/6.0)\]

Then, we compute the (scaled) membrane, $A/t^3$, bending, $D/t^3$, and shear, $S/t^3$, plate elastic stiffnesses:

\[A = \left(E*ts/t^3*(1. - nu**2)\right)\text{as_tensor}([[1., nu, 0.],[nu, 1., 0.],[0., 0., (1. - nu)/(-2)])]\]
\[D = \left(E*ts**3/t^3*(12.*((1. - nu**2))\right)\text{as_tensor}([[1., nu, 0.],[nu, 1., 0.],[0., 0., (1. - nu)/2])}\]
\[S = E*kappa*ts/t^3/(2*(1. + nu))\]

We use a $CG_2$ element for the in-plane and transverse displacements $u$ and $w$, and the enriched element $[CG_1 + B_3]$ for the rotations $\theta$. We collect the variables in the state vector $q = (u, w, \theta)$:

\[P1 = \text{FiniteElement}(\text{"Lagrange"}, \text{triangle}, \text{degree} = 1)\]
\[P2 = \text{FiniteElement}(\text{"Lagrange"}, \text{triangle}, \text{degree} = 2)\]
\[\text{bubble} = \text{FiniteElement}(\text{"B"}, \text{triangle}, \text{degree} = 3)\]
\[\text{enriched} = P1 + \text{bubble}\]
\[\text{element} = \text{MixedElement}([\text{VectorElement}(P2, \text{dim}=2), P2, \text{VectorElement}(\text{enriched}, \text{dim}=2)])\]
\[Q = \text{FunctionSpace}(\text{mesh}, \text{element})\]

Then, we define `Function`, `TrialFunction` and `TestFunction` objects to express the variational forms and we split them into each individual component function:

\[q, q, q_t = \text{Function}(Q), \text{TrialFunction}(Q), \text{TestFunction}(Q)\]
\[v, w, \theta, = \text{split}(q)\]

The membrane strain tensor $e$ for the von-Kármán plate takes into account the nonlinear contribution of the transverse displacement in the approximate form:

\[e(v, w) = \text{sym}\nabla v + \frac{\nabla w \otimes \nabla w}{2}\]

which can be expressed in UFL as:

\[e = \text{sym}(\text{grad}(v_)) + 0.5*\text{outer}(\text{grad}(w_), \text{grad}(w_))\]

The membrane energy density $\psi_m$ is a quadratic function of the membrane strain tensor $e$. For convenience, we use our function `strain_to_voigt()` to express $e$ in Voigt notation $e_V = \{e_1, e_2, 2e_{12}\}$:

\[ev = \text{strain_to_voigt}(e)\]
\[psi_m = 0.5*\text{dot}(A*ev, ev)\]

The bending strain tensor $k$ and shear strain vector $\gamma$ are identical to the standard Reissner-Mindlin model. The shear energy density $\psi_s$ is a quadratic function of the shear strain vector:

\[gamma = \text{grad}(w_) - \theta_,\]
\[psi_s = 0.5*\text{dot}(S*gamma, gamma)\]

The bending energy density $\psi_b$ is a quadratic function of the bending strain tensor. Here, the temperature profile on the plate is not modelled directly. Instead, it gives rise to an inelastic (initial) bending strain tensor $k_T$ which can be incorporated directly in the Lagrangian:
\[
\begin{align*}
\text{k}_T &= \text{as_tensor}\left(\text{Expression}\left(\left(\left(\left(c/\text{imp}, 0.0\right), \left(0.0, c*\text{imp}\right)\right)\text{)}\text{, c=1., imp=.999, degree=0}\text{)}\right)\right) \\
\text{k} &= \text{sym}(\text{grad} (\text{theta}_\text{a})) - \text{k}_T \\
\text{kv} &= \text{strain_to_voigt} (\text{k}) \\
\psi_b &= 0.5 * \text{dot} (\text{D} * \text{kv}, \text{kv})
\end{align*}
\]

**Note:** The standard von-Kármán model can be recovered by substituting in the Kirchoff constraint \( \theta = \nabla w \).

Shear- and membrane-locking are treated using the partial reduced selective integration proposed by Arnold and Brezzi, see [2]. In this approach shear and membrane energy are splitted as a sum of two contributions weighted by a factor \( \alpha \). One of the two contributions is integrated with a reduced integration. We use 2 × 2-points Gauss integration for a portion \( 1 - \alpha \) of the energy, whilst the rest is integrated with a 4 × 4 scheme. We adopt an optimized weighting factor \( \alpha = \left(\frac{t}{h}\right)^2 \), where \( h \) is the mesh size:

\[
\begin{align*}
\text{dx}_h &= \text{dx} (\text{metadata} = \{\text{'quadrature_degree': 2}\}) \\
\text{h} &= \text{CellDiameter} (\text{mesh}) \\
\text{alpha} &= \text{project} (\text{t} * 2 / \text{h} * 2, \text{FunctionSpace} (\text{mesh}, \text{'DG'}, 0)) \\
\text{Pi}_{PSRI} &= \psi_b * \text{dx} + \text{alpha} * \psi_m * \text{dx} + \alpha * \psi_s * \text{dx} + (1.0 - \text{alpha}) * \psi_s * \text{dx}_h + (1.0 - \alpha) * \psi_m * \text{dx}_h
\end{align*}
\]

Then, we compute the total elastic energy and its first and second derivatives:

\[
\begin{align*}
\text{Pi} &= \text{Pi}_{PSRI} \\
\text{dPi} &= \text{derivative} (\text{Pi}, \text{q}_t, \text{q}_t) \\
\text{J} &= \text{derivative} (\text{dPi}, \text{q}_t, \text{q}_t)
\end{align*}
\]

This problem is a pure Neumann problem. This leads to a nullspace in the solution. To remove this nullspace, we fix all the variables in the central point of the plate and the displacements in the \( x_0 \) and \( x_1 \) direction at \((0, a)\) and \((a, 0)\), respectively:

\[
\begin{align*}
\text{zero}_v1 &= \text{project} (\text{Constant} (\left(0.\right)), \text{Q}.\text{sub}(0).\text{sub}(0).\text{collapse}()) \\
\text{zero}_v2 &= \text{project} (\text{Constant} (\left(0.\right)), \text{Q}.\text{sub}(0).\text{sub}(1).\text{collapse}()) \\
\text{zero} &= \text{project} (\text{Constant} (\left(0., 0., 0., 0., 0.\right)), \text{Q}) \\
\text{bc} &= \text{DirichletBC} (\text{Q}, \text{zero}, \text{near} (\text{x}[0], 0.) \text{ and near} (\text{x}[1], 0.), \text{method} = \text{’pointwise’}) \\
\text{bc}_v1 &= \text{DirichletBC} (\text{Q}.\text{sub}(0).\text{sub}(0), \text{zero}_v1, \text{near} (\text{x}[0], 0.) \text{ and near} (\text{x}[1], 1.), \text{method} = \text{’pointwise’}) \\
\text{bc}_v2 &= \text{DirichletBC} (\text{Q}.\text{sub}(0).\text{sub}(1), \text{zero}_v2, \text{near} (\text{x}[0], 1.) \text{ and near} (\text{x}[1], 0.), \text{method} = \text{’pointwise’}) \\
\text{bcs} &= [\text{bc}, \text{bc}_v1, \text{bc}_v2]
\end{align*}
\]

Then, we define the nonlinear variational problem and the solver settings:

\[
\begin{align*}
\text{init} &= \text{Function} (\text{Q}) \\
\text{q}_t &= \text{assign} (\text{init}) \\
\text{problem} &= \text{NonlinearVariationalProblem} (\text{dPi}, \text{q}_t, \text{bcs}, \text{J} = \text{J}) \\
\text{solver} &= \text{NonlinearVariationalSolver} (\text{problem}) \\
\text{solver.parameters}[\text{’newton_solver’}][\text{’absolute_tolerance’}] &= 1E-8
\end{align*}
\]

Finally, we choose the continuation steps (the critical loading \( \text{c}_{cr} \) is taken from the Mansfield analytical solution [1]):

\[
\begin{align*}
\text{c}_{cr} &= 0.0516 \\
\text{cs} &= \text{np.linspace} (0.0, 1.5 * \text{c}_{cr}, 30)
\end{align*}
\]
and we solve as usual:

```python
def plots_dir = "output/3dplots-psri/
file = File(plots_dir + "sol.pvd")
ls_kx = []
ls_ky = []
ls_kxy = []
ls_kT = []

for count, i in enumerate(cs):
k_T.c = i
  solver.solve()
v_h, w_h, theta_h = q_.split(deepcopy=True)

To visualise the solution we assemble the bending strain tensor:

```python
K_h = project(sym(grad(theta_h)), TensorFunctionSpace(mesh, 'DG', 0))
```

we compute the average bending strain:

```python
Kxx = assemble(K_h[0,0]*dx)/domain_area
Kyy = assemble(K_h[1,1]*dx)/domain_area
Kxy = assemble(K_h[0,1]*dx)/domain_area
ls_kT.append(i)
ls_kx.append(Kxx)
ls_ky.append(Kyy)
ls_kxy.append(Kxy)
```

and output the results at each continuation step:

```python
v1_h, v2_h = v_h.split()
u_h = as_vector([v1_h, v2_h, w_h])
u_h_pro = project(u_h, VectorFunctionSpace(mesh, 'CG', 1, dim=3))
file << u_h_pro
```

Finally, we plot the average curvatures as a function of the inelastic curvature:

```python
fig = plt.figure(figsize=(5.0, 5.0/1.648))
plt.plot(ls_kT, ls_kx, "o", color='orange', label=r"$k_{1h}$")
plt.plot(ls_kT, ls_ky, "x", color='red', label=r"$k_{2h}$")
plt.xlabel(r"inelastic curvature $\eta$")
plt.ylabel(r"curvature $k_{1,2}$")
plt.legend()
plt.tight_layout()
plt.savefig("psri-%s.png"%ndiv)
```

Unit testing

```python
import pytest

@pytest.mark.skip
def test_close():
  pass
```
3.2.5 Bifurcation of a composite laminate plate modeled with von-Kármán theory

This demo is implemented in the single Python file demo_von-karman-composite.py.

This demo program solves the von-Kármán equations for an elliptic composite plate with a lenticular cross section free on the boundary. The plate is heated, causing it to bifurcate. Bifurcation occurs with a striking shape transition: before the critical threshold the plate assumes a cup-shaped configuration (left); above it tends to a cylindrical shape (right).

An analytical solution can be computed using the procedure outlined in the paper [Maurini].

The standard von-Kármán theory gives rise to a fourth-order PDE which requires the transverse displacement field $w$ to be sought in the space $H^2(\Omega)$. We relax this requirement in the same manner as the Kirchoff-Love plate theory can be relaxed to the Reissner-Mindlin theory. Accordingly, we seek a transverse displacement field $w$ in $H^1(\Omega)$ and a rotation field $\theta$ in $[H^2(\Omega)]^2$. To alleviate the resulting shear-locking issue we then apply the Durán-Liberman reduction operator.

To follow this demo you should know how to:

- Define a MixedElement and a FunctionSpace from it.
- Define the Durán-Liberman (MITC) reduction operator using UFL. This procedure eliminates the shear-locking problem.
- Write variational forms using the Unified Form Language.
- Use the fenics-shell function laminate() to compute the stiffness matrices.
• Automatically derive Jacobian and residuals using \texttt{derivative}().
• Apply Dirichlet boundary conditions using \texttt{DirichletBC} and \texttt{apply}().
• Assemble forms using \texttt{assemble}().
• Solve linear systems using \texttt{LUSolver}.
• Output data to XDMF files with \texttt{XDMFFile}.

This demo then illustrates how to:
• Define the Reissner-Mindlin-von-Kármán plate equations using UFL.
• Use the \texttt{ProjectedNonlinearSolver} class to drive the solution of a non-linear problem with projected variables.

We begin by setting up our Python environment with the required modules:

```python
from dolfin import *
from ufl import RestrictedElement

from fenics_shells import *
import numpy as np

try:
    import matplotlib.pyplot as plt
except ImportError:
    raise ImportError("matplotlib is required to run this demo.")

try:
    import mshr
except ImportError:
    raise ImportError("mshr is required to run this demo.")
```

The mid-plane of the plate is an elliptic domain with semi-axes $a = 1$ and $b = 0.5$. We generate a mesh of the domain using the package \texttt{mshr}:
a_rad = 1.0
b_rad = 0.5
n_div = 30

centre = Point(0., 0.)

geom = mshr.Ellipse(centre, a_rad, b_rad)
mesh = mshr.generate_mesh(geom, n_div)

The lenticular thinning of the plate can be modelled directly through the thickness parameter in the plate model:

```python
h = interpolate(Expression('t0*(1.0 - (x[0]*x[0])/(a*a) - (x[1]*x[1])/(b*b))', t0=1E-2, a=a_rad, b=b_rad, degree=2), FunctionSpace(mesh, 'CG', 2))
```

We assume the plate is a composite laminate with 8-layer stacking sequence:

\[ [45^\circ, -45^\circ, 45^\circ, 45^\circ, 45^\circ, -45^\circ, 45^\circ, -45^\circ] \]

with elementary layer properties \( E_1 = 40.038, E_2 = 1.0, G_{12} = 0.5, \nu_{12} = 0.25, G_{23} = 0.4 \):

```python
E1 = 40.038
E2 = 1.0
G12 = 0.5
nu12 = 0.25
G23 = 0.4
```

We use our function `laminates()` to compute the stiffness matrices according to the Classical Laminate Theory:

```python
n_layers = len(thetas)
hs = h*np.ones(n_layers)/n_layers
A, B, D = laminates.ABD(E1, E2, G12, nu12, hs, thetas)
Fs = laminates.F(G12, G23, hs, thetas)
```

We then define our MixedElement which will discretise the in-plane displacements \( v \in [CG_1]^2 \), rotations \( \theta \in [CG_2]^2 \), out-of-plane displacements \( w \in CG_1 \). Two further auxiliary fields are also considered, the reduced shear strain \( \gamma_R \), and a Lagrange multiplier field \( p \) which ties together the shear strain calculated from the primal variables \( \gamma = \nabla w - \theta \) and the reduced shear strain \( \gamma_R \). Both \( p \) and \( \gamma_R \) are discretised in the space \( NED_1 \), the vector-valued Nédélec elements of the first kind. The final element definition is then:

```python
element = MixedElement([VectorElement("Lagrange", triangle, 1),
    VectorElement("Lagrange", triangle, 2),
    FiniteElement("Lagrange", triangle, 1),
    FiniteElement("N1curl", triangle, 1),
    RestrictedElement(FiniteElement("N1curl", triangle, 1), "edge")])
```

We then pass our `element` through to the `ProjectedFunctionSpace` constructor. As we will see later in this example, we can project out both the \( p \) and \( \gamma_R \) fields at assembly time. We specify this by passing the argument `num_projected_subspaces=2`:

```python
U = ProjectedFunctionSpace(mesh, element, num_projected_subspaces=2)
U_F = U.full_space
U_P = U.projected_space
```

Using only the `full` function space object `U_F` we setup our variational problem by defining the Lagrangian of our problem. We begin by creating a `Function` and splitting it into each individual component function:

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The membrane strain tensor $\epsilon$ for the von-Kármán plate takes into account the nonlinear contribution of the transverse displacement in the approximate form:

$$
\epsilon(v, w) = \text{sym}(\nabla v) + \frac{\nabla w \otimes \nabla w}{2}
$$

which can be expressed in UFL as:

$$
e = \text{sym}(\nabla(v_\text{r})) + 0.5 \times \text{outer}(\nabla(w_\text{r}), \nabla(w_\text{r}))
$$

The membrane energy density $\psi_N$ is a quadratic function of the membrane strain tensor $\epsilon$. For convenience, we use our function $\text{strain} \_ \text{to} \_ \text{voigt()}$ to express $\epsilon$ in Voigt notation $e_V = \{e_1, e_2, 2e_{12}\}$:

$$
ev = \text{strain} \_ \text{to} \_ \text{voigt}(e)
\text{Ai} = \text{project}(A, \text{TensorFunctionSpace(mesh, 'CG', 1, shape=(3,3)))}
\psi_N = 0.5 \times \text{dot}(\text{Ai} \times ev, ev)
$$

The bending strain tensor $k$ and shear strain vector $\gamma$ are identical to the standard Reissner-Mindlin model. The shear energy density $\psi_T$ is a quadratic function of the reduced shear vector:

$$
\text{Fi} = \text{project}(F_s, \text{TensorFunctionSpace(mesh, 'CG', 1, shape=(2,2)))}
\psi_T = 0.5 \times \text{dot}(\text{Fi} \times R\_\gamma, R\_\gamma)
$$

The bending energy density $\psi_M$ is a quadratic function of the bending strain tensor. Here, the temperature profile on the plate is not modelled directly. Instead, it gives rise to an inelastic (initial) bending strain tensor $k_T$ which can be incorporated directly in the Lagrangian:

$$
k_T = \text{as_tensor}(\text{Expression}(("1.0*c","0.0*c"),("0.0*c","1.0*c")), c=1.0, \text{degree}=0)
k = \text{sym}(\nabla\theta_\text{r}) - k_T
\text{kv} = \text{strain} \_ \text{to} \_ \text{voigt}(k)
\text{Di} = \text{project}(D, \text{TensorFunctionSpace(mesh, 'CG', 1, shape=(3,3)))}
\psi_M = 0.5 \times \text{dot}(\text{Di} \times kv, kv)
$$

Note: The standard von-Kármán model can be recovered by substituting in the Kirchoff constraint $\theta = \nabla w$.

Finally, we define the membrane-bending coupling energy density $\psi_{NM}$, even if it vanishes in this case:

$$
\text{Bi} = \text{project}(B, \text{TensorFunctionSpace(mesh, 'CG', 1, shape=(3,3)))}
\psi_{MN} = \text{dot}(\text{Bi} \times kv, ev)
$$

This problem is a pure Neumann problem. This leads to a nullspace in the solution. To remove this nullspace, we fix the displacements in the central point of the plate:

```python
h_max = mesh.hmax()
def center(x, on_boundary):
    return x[0]**2 + x[1]**2 < (0.5*h_max)**2
bc_v = DirichletBC(U.sub(0), Constant((0.0,0.0)), center, method="pointwise")
bc_R = DirichletBC(U.sub(1), Constant((0.0,0.0)), center, method="pointwise")
bc_w = DirichletBC(U.sub(2), Constant(0.0), center, method="pointwise")
```
Finally, we define the Durán-Liberman reduction operator by tying the shear strain calculated with the displacement variables $\gamma = \nabla w - \theta$ to the conforming reduced shear strain $\gamma_R$ using the Lagrange multiplier field $p$:

```
\gamma = \text{grad}(w_) - \theta_
L_R = \text{inner}_e(\gamma - R_{\gamma_\cdot}, p_)
```

We can now define our Lagrangian for the complete system:

```
L = (psi_M + psi_T + psi_N + psi_MN)*dx + L_R
F = \text{derivative}(L, u_\cdot, u_t)
J = \text{derivative}(F, u_\cdot, u)
```

The solution of a non-linear problem with the `ProjectedFunctionSpace` functionality is a little bit more involved than the linear case. We provide a special class `ProjectedNonlinearProblem` which conforms to the DOLFIN `NonlinearProblem` interface that hides much of the complexity.

**Note:** Inside `ProjectedNonlinearProblem`, the Jacobian and residual equations on the projected space are assembled using the special assembler in FEniCS Shells. The Newton update is calculated on the space $U_P$. Then, it is necessary to update the variables in the full space $U_F$ before performing the next Newton iteration.

In practice, the interface is nearly identical to a standard implementation of `NonlinearProblem`, except the requirement to pass a `Function` on both the full $u_\cdot$ and projected spaces $u_p_\cdot$:

```
u_p_ = \text{Function}(U_P)
problem = ProjectedNonlinearProblem(U_P, F, u_, u_p_, bcs=bcs, J=J)
solver = NewtonSolver()
solver.parameters[\text{'absolute tolerance'] = 1E-12
```

We apply the inelastic curvature with 20 continuation steps. The critical loading $c_{\text{cr}}$ as well as the solution in terms of curvatures is taken from the analytical solution. Here, $R_0$ is the scaling radius of curvature and $\beta = A_{2222}/A_{1111} = D_{2222}/D_{1111}$ as in [Maurini]:

```
from fenics_shells.analytical.vonkarman_heated import analytical_solution
c_cr, beta, R0, h_before, h_after, ls_Kbefore, ls_K1after, ls_K2after = analytical_solution(Ai, Di, a_rad, b_rad)
cs = np.linspace(0.0, 1.5*c_cr, 20)
```

We solve as usual:

```
domain_area = np.pi*a_rad*b_rad
kx = []
ky = []
kxy = []
ls_load = []

for i, c in enumerate(cs):
    k_T.c = c
    solver.solve(problem, u_p_.vector())
    v_h, theta_h, w_h, R_theta_h, p_h = u_.split()
```

Then, we assemble the bending strain tensor:

```
k_h = \text{sym}(\text{grad}(theta_h))
K_h = \text{project}(k_h, \text{TensorFunctionSpace}(mesh, 'DG', 0))
```

we calculate the average bending strain:

---

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\[
K_{xx} = \text{assemble}(K_h[0,0] \cdot dx)/\text{domain\_area}
\]
\[
K_{yy} = \text{assemble}(K_h[1,1] \cdot dx)/\text{domain\_area}
\]
\[
K_{xy} = \text{assemble}(K_h[0,1] \cdot dx)/\text{domain\_area}
\]

\[
\text{ls\_load}.append(c*R0)
\]
\[
\text{kx}.append(K_{xx}*R0/\text{np}\sqrt{\beta})
\]
\[
\text{ky}.append(K_{yy})
\]
\[
\text{kxy}.append(K_{xy}*R0/(\beta^{1.0/4.0}))
\]

and output the results at each continuation step:

```python
save_dir = "output/
fields = {"theta": theta_h, "\nu": v_h, "\omega": w_h, "R\_theta": R\_theta_h}
for name, field in fields.items():
    field.rename(name, name)
    field_file = XDMFFile("\{}
```

Finally, we compare numerical and analytical solutions:

```python
fig = plt.figure(figsize=(5.0, 5.0/1.648))
plt.plot(ls_load, kx, "o", color='r', label=r"$k_{1h}$")
plt.plot(ls_load, ky, "x", color='green', label=r"$k_{2h}$")
plt.plot(h\_before, ls\_Kbefore, "-", color='b', label="Analytical solution")
plt.plot(h\_after, ls\_K1after, "-", color='b')
plt.plot(h\_after, ls\_K2after, "-", color='b')
plt.xlabel(r"inelastic curvature $\eta$")
plt.ylabel(r"curvature $k_{1,2}$")
plt.legend()
plt.tight\_layout()
plt.savefig("curvature\_bifurcation.png")
```
Unit testing

```python
import pytest
@ pytest.mark.skip
def test_close():
    pass
```

Linear and non-linear Naghdi shell problem solved with MITC and PRSI locking cure

### 3.2.6 Partly Clamped Hyperbolic Paraboloid

This demo is implemented in the single Python file `demo_naghdi-linear-hypar.py`.

This demo program solves the linear Naghdi shell equations for a partly-clamped hyperbolic paraboloid (hypar) subjected to a uniform vertical loading. This is a well known bending dominated benchmark problem for testing a FE formulation with respect to membrane locking issues, see [1]. Here, locking is cured using enriched finite elements including cubic bubble shape functions and Partial Selective Reduced Integration (PSRI), see [2, 3].

To follow this demo you should know how to:

- Define a MixedElement and EnrichedElement and a FunctionSpace from it.
- Write variational forms using the Unified Form Language.
- Automatically derive Jacobian and residuals using derivative().
- Apply Dirichlet boundary conditions using DirichletBC and apply().

This demo then illustrates how to:

- Define and solve a linear Naghdi shell problem with a curved stress-free configuration given as analytical expression in terms of two curvilinear coordinates.
We start with importing the required modules, setting `matplotlib` as plotting backend, and generically set the integration order to 4 to avoid the automatic setting of FEniCS which would lead to unreasonably high integration orders for complex forms:

```python
import os, sys
import numpy as np
import matplotlib.pyplot as plt
from dolfin import *
from mshr import *
from ufl import Index
from mpl_toolkits.mplot3d import Axes3D

parameters["form_compiler"]['quadrature_degree'] = 4
output_dir = "output/"
if not os.path.exists(output_dir):
    os.makedirs(output_dir)
```

We consider an hyperbolic paraboloid shell made of a linear elastic isotropic homogeneous material with Young modulus \( Y \) and Poisson ratio \( \nu \); \( \mu \) and \( \lambda \) denote the shear modulus \( \mu = \frac{Y}{2(1 + \nu)} \) and the Lamé constant \( \lambda = \frac{2\mu\nu}{(1-2\nu)} \). The (uniform) shell thickness is denoted by \( t \).

```python
Y, nu = 2.0e8, 0.3
mu = Y/(2.0*(1.0 + nu))
lb = 2.0*mu*nu/(1.0 - 2.0*nu)
t = Constant(1E-2)
```

The midplane of the initial (stress-free) configuration \( S_I \) of the shell is given in the form of an analytical expression

\[
y_I : x \in \mathcal{M} \subset \mathbb{R}^2 \rightarrow y_I(x) \in S_I \subset \mathbb{R}^3 \quad S_I = \{x_1, x_2, x_1^2 - x_2^2\}
\]

in terms of the curvilinear coordinates \( x \). Hence, we mesh the two-dimensional domain \( \mathcal{M} \equiv [-L/2, L/2] \times [-L/2, L/2] \):

```python
L = 1.0
P1, P2 = Point(-L/2, -L/2), Point(L/2, L/2)
div = 40
mesh = RectangleMesh(P1, P2, div, div)
```

We provide the analytical expression of the initial shape as an `Expression` that we represent on a suitable `FunctionSpace` (here \( P_2 \), but other are choices are possible):

```python
initial_shape = Expression(('x[0]','x[1]','x[0]*x[0] - x[1]*x[1]'), degree = 4)
V_y = FunctionSpace(mesh, VectorElement("P", triangle, degree=2, dim=3))
yI = project(initial_shape, V_y)
```

We compute the covariant and contravariant component of the metric tensor, \( a_I, a_I\_contra \); the covariant base vectors \( g_0, g_1 \); the contravariant base vectors \( g_0\_c, g_1\_c \).

```python
aI = grad(yI).T*grad(yI)
aI\_contra, jI = inv(aI), det(aI)
g0, g1 = yI.dx(0), yI.dx(1)
g0\_c, g1\_c = aI\_contra[0,0]*g0 + aI\_contra[0,1]*g1, aI\_contra[1,0]*g0 + aI\_contra[1, 1]*g1
```
Given the midplane, we define the corresponding unit normal as below and project on a suitable function space (here $P_1$ but other choices are possible):

```python
def normal(y):
    n = cross(y.dx(0), y.dx(1))
    return n/sqrt(inner(n,n))
V_normal = FunctionSpace(mesh, VectorElement("P", triangle, degree = 1, dim = 3))
nI = project(normal(yI), V_normal)
```

We can visualize the shell shape and its normal with this utility function

```python
def plot_shell(y, n=None):
    y_0, y_1, y_2 = y.split(deepcopy=True)
    fig = plt.figure()
    ax = fig.gca(projection='3d')
    ax.plot_trisurf(y_0.compute_vertex_values(),
                    y_1.compute_vertex_values(),
                    y_2.compute_vertex_values(),
                    triangles=y.function_space().mesh().cells(),
                    linewidth=1, antialiased=True, shade=False)
    if n:
        n_0, n_1, n_2 = n.split(deepcopy=True)
        ax.quiver(y_0.compute_vertex_values(),
                  y_1.compute_vertex_values(),
                  y_2.compute_vertex_values(),
                  n_0.compute_vertex_values(),
                  n_1.compute_vertex_values(),
                  n_2.compute_vertex_values(),
                  length=.2, color="r")
    ax.view_init(elev=50, azim=30)
    ax.set_xlim(-0.5, 0.5)
    ax.set_ylim(-0.5, 0.5)
    ax.set_zlim(-0.5, 0.5)
    ax.set_xlabel(r"$x_0$"
    ax.set_ylabel(r"$x_1$"
    ax.set_zlabel(r"$x_2$"
    return ax
plot_shell(yI)
plt.savefig("output/initial_configuration.png")
```

In our 5-parameter Naghdi shell model the configuration of the shell is assigned by
- the 3-component vector field $u_\cdot$ representing the (small) displacement with respect to the initial configuration $y_I$
- the 2-component vector field $\theta_\cdot$ representing the (small) rotation of fibers orthogonal to the middle surface.

Following [2, 3], we use a $P2+bubble$ element for $u_\cdot$ and a $P2$ element for $\theta_\cdot$, and collect them in the state vector $z_\cdot=[u_\cdot, \theta_\cdot]$. We further define `Function`, `TestFunction`, and `TrialFunction` and their different split views, which are useful for expressing the variational formulation.

```python
P2 = FiniteElement("P", triangle, degree = 2)
bubble = FiniteElement("B", triangle, degree = 3)
Z = FunctionSpace(mesh, MixedElement(3*[P2 + bubble ] + 2*[P2]))
z_\cdot = Function(Z)
```
We define the displacement vector and the rotation vector, with this latter tangent to the middle surface, $\theta = \theta_\sigma g^\sigma$, $\sigma = 0, 1$

$$u_\sigma, u, ut = \text{as_vector}([u_0, u_1, u_2]), \text{as_vector}([u_0, u_1, u_2]), \text{as_vector}([u_0t, ut, -u_2t])$$

$$\theta_\sigma, \theta, \theta t = th0_\sigma g0_c + th1_\sigma g1_c, th0 g0_c + th1 g1_c, th0t g0_c + th1t g1_c \rightarrow thlt g1_c$$

The extensional, $e_naghdi$, bending, $k_naghdi$, and shearing, $g_naghdi$, strains in the linear Naghdi model are defined by

$$e_naghdi = \text{lambda}(v) : 0.5*(\text{grad}(yI).T*\text{grad}(v) + \text{grad}(v).T*\text{grad}(yI))$$

$$k_naghdi = \text{lambda}(v, t) : -0.5*(\text{grad}(yI).T*\text{grad}(t) + \text{grad}(t).T*\text{grad}(yI)) - 0.5*(\text{grad}(nI).T*\text{grad}(v) + \text{grad}(v).T*\text{grad}(nI))$$

$$g_naghdi = \text{lambda}(v, t) : \text{grad}(yI).T*t + \text{grad}(v).T*nI$$

Using curvilinear coordinates, the constitutive equations are written in terms of the matrix $A_hooke$ below, representing the contravariant components of the constitutive tensor for isotropic elasticity in plane stress, see e.g. [4]. We use the index notation offered by UFL to express operations between tensors

$$i, j, k, l = \text{Index}(), \text{Index}(), \text{Index}(), \text{Index}()$$

$$A_hooke = \text{as_tensor}(((2.0*lb*mu)/(lb + 2.0*mu))*aI_contra[i,j]*aI_contra[k,l] + 1.0*mu*(aI_contra[i,k]*aI_contra[j,l] + aI_contra[i,l]*aI_contra[j,k])),[i,j,k,l])$$

The membrane stress and bending moment tensors, $N$ and $M$, and shear stress vector, $T$, are

$$N = \text{as_tensor}((A_hooke[i,j,k,l]*e_naghdi(u_)[k,l]),[i, j])$$

$$M = \text{as_tensor}(((1./12.0)*A_hooke[i,j,k,l]*k_naghdi(u_, theta_)[k,l]),[i, j])$$

$$T = \text{as_tensor}((mu*aI_contra[i,j]*g_naghdi(u_, theta_)[j]), [i])$$

Hence, the contributions to the elastic energy densities due to membrane, $\psi_m$, bending, $\psi_b$, and shear, $\psi_s$, are

$$\psi_m = 0.5*\text{inner}(N, e_naghdi(u_))$$

$$\psi_b = 0.5*\text{inner}(M, k_naghdi(u_, theta_))$$

$$\psi_s = 0.5*\text{inner}(T, g_naghdi(u_, theta_))$$

Shear and membrane locking are treated using the PSRI proposed by Arnold and Brezzi, see [2, 3]. In this approach, shear and membrane energies are split as a sum of two weighted contributions, one of which is computed with a reduced integration. Thus, shear and membrane energies have the form

$$(i = m, s) \quad \alpha \int_M \psi_i \sqrt{J} d\mathbf{x} + (\kappa - \alpha) \int_M \psi_i \sqrt{J} d\mathbf{x}_h, \quad \text{where } \kappa \propto t^{-2}$$

While [2, 3] suggest a 1-point reduced integration, we observed that this leads to spurious modes in the present case. We use then $2 \times 2$-points Gauss integration for the portion $\kappa - \alpha$ of the energy, whilst the rest is integrated with a $4 \times 4$ scheme. As suggested in [3], we adopt an optimized weighting factor $\alpha = 1$

$$dx_h = dx(\text{metadata}={'\text{quadrature_degree}': 2})$$

alpha = 1.0

kappa = 1.0/t**2
shear_energy = alpha*psi_s*sqrt(jI)*dx + (kappa - alpha)*psi_s*sqrt(jI)*dx_h
membrane_energy = alpha*psi_m*sqrt(jI)*dx + (kappa - alpha)*psi_m*sqrt(jI)*dx_h
bending_energy = psi_b*sqrt(jI)*dx

Then, the elastic energy is

elastic_energy = (t**3)*(bending_energy + membrane_energy + shear_energy)

The shell is subjected to a constant vertical load. Thus, the external work is

body_force = 8.*t
f = Constant(body_force)
external_work = f*u2_*sqrt(jI)*dx

We now compute the total potential energy with its first and second derivatives

Pi_total = elastic_energy - external_work
residual = derivative(Pi_total, z_, zt)
hessian = derivative(residual, z_, z)

The boundary conditions prescribe a full clamping on the \( x_0 = -L/2 \) boundary, while the other sides are left free

left_boundary = lambda x, on_boundary: abs(x[0] + L/2) <= DOLFIN_EPS and on_boundary
clamp = DirichletBC(Z, project(Expression(('0.0', '0.0', '0.0', '0.0', '0.0'), degree=1), Z), left_boundary)
bcs = [clamp]

We now solve the linear system of equations

output_dir = "output/"
A, b = assemble_system(hessian, residual, bcs=bcs)
solver = PETScLUSolver("mumps")
solver.solve(A, z_.vector(), b)
u0_h, u1_h, u2_h, th0_h, th1_h = z_.split(deepcopy=True)

Finally, we can plot the final configuration of the shell

scale_factor = 1e4
plot_shell(project(scale_factor*u_ + yI, V_y))
plt.savefig("output/finalconfiguration.png")

References

3.2.7 A non-linear Naghdi roll-up cantilever

This demo is implemented in the single Python file demo_nonlinear-naghdi-cantilever.py.

This demo program solves the non-linear Naghdi shell equations on a rectangular plate with a constant bending moment applied. The plate rolls up completely on itself. The numerical locking issue is cured using a Durán-Liberman approach.

To follow this demo you should know how to:

- Define a MixedElement and a FunctionSpace from it.
- Define the Durán-Liberman (MITC) reduction operator using UFL for a linear problem, e.g. Reissner-Mindlin. This procedure extends simply to the non-linear problem we consider here.
- Write variational forms using the Unified Form Language.
- Automatically derive Jacobian and residuals using derivative().
- Apply Dirichlet boundary conditions using DirichletBC and apply().
- Apply Neumann boundary conditions by marking a FacetFunction and create a new Measure object.
- Solve non-linear problems using ProjectedNonlinearProblem.
- Output data to XDMF files with XDMFFile.

This demo then illustrates how to:

- Define and solve a non-linear Naghdi shell problem with a flat reference configuration.

We begin by setting up our Python environment with the required modules:

```python
import os
import numpy as np
import matplotlib.pyplot as plt
from dolfin import *
from ufl import RestrictedElement
from fenics_shells import *
```

We set the default quadrature degree. UFL's built in quadrature degree detection often overestimates the required degree for these complicated forms:

```python
parameters["form_compiler"]['quadrature_degree'] = 2
```

Our reference middle surface is a rectangle \( \omega = [0, 12] \times [-0.5, 0.5] \):

```python
length = 12.0
width = 1.0
P1, P2 = Point(0.0, -width/2.0), Point(length, width/2.0)
mesh = RectangleMesh(P1, P2, 48, 4, "crossed")
```

We then define our MixedElement which will discretise the in-plane displacements \( v \in [CG_1]^2 \), rotations \( \beta \in [CG_2]^2 \), out-of-plane displacements \( w \in CG_1 \), the shear strains. Two further auxiliary fields are also considered, the reduced shear strain \( \gamma_R \), and a Lagrange multiplier field \( p \) which ties together the Naghdi shear strain calculated from the primal variables and the reduced shear strain \( \gamma_R \). Both \( p \) and \( \gamma_R \) are are discretised in the space \( NED_1 \), the vector-valued Nédélec elements of the first kind. The final element definition is then:

```python
element = MixedElement([
    VectorElement("Lagrange", triangle, 1),
    VectorElement("Lagrange", triangle, 2),
    FiniteElement("Lagrange", triangle, 1),
])
```
We then pass our element through to the ProjectedFunctionSpace constructor. As in the other documented demos, we can project out $p$ and $\gamma_R$ fields at assembly time. We specify this by passing the argument num_projected_subspaces=2:

```
U = ProjectedFunctionSpace(mesh, element, num_projected_subspaces=2)
U_F = U.full_space
U_P = U.projected_space
```

We assume constant material parameters; Young’s modulus $E$, Poisson’s ratio $\nu$, and thickness $t$:

```
E, nu = Constant(1.2E6), Constant(0.0)
mu = E/(2.0*(1.0 + nu))
lmbda = 2.0*mu*nu/(1.0 - 2.0*nu)
t = Constant(1E-1)
```

Using only the full function space object $U_F$ we setup our variational problem by defining the Lagrangian of our problem. We begin by creating a Function and splitting it into each individual component function:

```
u, u_t, u_ = TrialFunction(U_F), TestFunction(U_F), Function(U_F)
v_, beta_, w_, Rgamma_, p_ = split(u_)
```

For the Naghdi problem it is convenient to recombine the in-plane displacements $v$ and out-of-plane displacements $w$ into a single vector field $z$:

```
z_ = as_vector([v_[0], v_[1], w_])
```

We can now define our non-linear Naghdi strain measures. Assuming the normal fibres of the shell are unstrechable, we can parameterise the director vector field $d: \omega \rightarrow \mathbb{R}^3$ using the two independent rotations $\beta$:

```
d = as_vector([sin(beta_[1])*cos(beta_[0]), -sin(beta_[0]), cos(beta_[1])*cos(beta_→[0])])
```

The deformation gradient $F$ can be defined as:

```
F = grad(z_) + as_tensor([[1.0, 0.0],
                         [0.0, 1.0],
                         [Constant(0.0), Constant(0.0)])
```

From which we can define the stretching (membrane) strain $e$:

```
e = 0.5*(F.T*F - Identity(2))
```

The curvature (bending) strain $k$:

```
k = 0.5*(F.T*grad(d) + grad(d).T*F)
```

and the shear strain $\gamma$:

```
gamma = F.T*d
```

We then define the constitutive law in terms of a general dual strain measure tensor $X$: 

```
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```
from which we can define the membrane energy density:

\[
\psi_N = 0.5 \times t \times \text{inner}(S(e), e)
\]

the bending energy density:

\[
\psi_K = 0.5 \times \left(\frac{t \times 3}{12.0}\right) \times \text{inner}(S(k), k)
\]

and the shear energy density:

\[
\psi_T = 0.5 \times t \times \mu \times \text{inner}(\taugamma_, \taugamma_)
\]

and the total energy density from all three contributions:

\[
\psi = \psi_N + \psi_K + \psi_T
\]

We define the Durán-Liberman reduction operator by tying the shear strain calculated with the displacement variables \( \gamma = F^T d \) to the reduced shear strain \( \gamma_R \) using the Lagrange multiplier field \( p \):

\[
L_R = \text{inner}_e(\gamma - \gamma_R, p_)
\]

We then turn to defining the boundary conditions and external loading. On the left edge of the domain we apply clamped boundary conditions which corresponds to constraining all generalised displacement fields to zero:

\[
\text{left} = \lambda x, \text{on_boundary}: x[0] \leq \text{DOLFIN_EPS and on_boundary}
\]

\[
\text{bc}_v = \text{DirichletBC(U.sub(0), Constant((0.0, 0.0)), left)}
\]

\[
\text{bc}_a = \text{DirichletBC(U.sub(1), Constant((0.0, 0.0)), left)}
\]

\[
\text{bc}_w = \text{DirichletBC(U.sub(2), Constant(0.0), left)}
\]

\[
bcs = [\text{bc}_v, \text{bc}_a, \text{bc}_w]
\]

On the right edge of the domain we apply a traction:

\[
\# \text{Define subdomain for boundary condition on tractions}
\]

```python
class Right(SubDomain):
    def inside(self, x, on_boundary):
        return abs(x[0] - 12.0) <= DOLFIN_EPS and on_boundary

right_tractions = Right()
```

\[
\text{exterior}\_\text{facet}\_\text{domains} = \text{MeshFunction("size_t", mesh, mesh.geometry().dim() - 1)}
\]

\[
\text{ds} = \text{Measure("ds", subdomain_data=exterior\_\text{facet}\_\text{domains})}
\]

\[
M\_\text{right} = \text{Expression\{"M\}}, M=0.0, \text{degree}=0
\]

\[
W\_\text{ext} = M\_\text{right} + \text{beta\_}[1] \times ds[1]
\]

We can now define our Lagrangian for the complete system:

\[
L = \psi\_dx + L\_R - W\_\text{ext}
\]

\[
F = \text{derivative}(L, u\_-, u\_t)
\]

\[
J = \text{derivative}(F, u\_-, u)
\]

Before setting up the non-linear problem with the special \textit{ProjectedFunctionSpace} functionality:
u_p_ = Function(U_P)

problem = ProjectedNonlinearProblem(U_P, F, u_, u_p_, bcs=bcs, J=J)
solver = NewtonSolver()

and solving:

solver.parameters['error_on_nonconvergence'] = False
solver.parameters['maximum_iterations'] = 20
solver.parameters['linear_solver'] = 'mumps'
solver.parameters['absolute_tolerance'] = 1E-20
solver.parameters['relative_tolerance'] = 1E-6

output_dir = "output/
if not os.path.exists(output_dir):
    os.makedirs(output_dir)

We apply the moment with 20 continuation steps:

M_max = 2.0*np.pi*E.values()[0]*t.values()[0]**3/(12.0*length)
Ms = np.linspace(0.0, M_max, 20)

w_hs = []
v_hs = []

for i, M in enumerate(Ms):
    M_right.M = M
    solver.solve(problem, u_p_.vector())
    v_h, theta_h, w_h, Rgamma_h, p_h = u_.split(deepcopy=True)
    z_h = project(z_, VectorFunctionSpace(mesh, "CG", 1, dim=3))
    z_h.rename('z', 'z')
    XDMFFile(output_dir + "z_{}/xdmf".format(str(i).zfill(3))).write(z_h)
    w_hs.append(w_h(length, 0.0))
    v_hs.append(v_h(length, 0.0)[0])

This problem has a simple closed-form analytical solution which we plot against for comparison:

w_hs = np.array(w_hs)
v_hs = np.array(v_hs)

Ms_analytical = np.linspace(1E-3, 1.0, 100)
vs = 12.0*(np.sin(2.0*np.pi*Ms_analytical)/(2.0*np.pi*Ms_analytical) - 1.0)
ws = -12.0*(1.0 - np.cos(2.0*np.pi*Ms_analytical))/(2.0*np.pi*Ms_analytical)

fig = plt.figure(figsize=(5.0, 5.0/1.648))
plt.plot(Ms_analytical, vs/length, "--", label="$v/L$")
plt.plot(Ms/M_max, v_hs/length, "o", label="$v_h/L$")
plt.plot(Ms_analytical, ws/length, "-", label="$w/L$")
plt.plot(Ms/M_max, w_hs/length, "x", label="$w_h/L$")
plt.xlabel("$M/M_{\text{max}}$")
plt.ylabel("normalised displacement")
plt.legend()
plt.tight_layout()
plt.savefig("output/cantilever-displacement-plot.pdf")
plt.savefig("output/cantilever-displacement-plot.png")
def test_close():
    assert (np.isclose(w_h(length, 0.0)/length, 0.0, atol=1E-3, rtol=1E-3))
    assert (np.isclose(v_h(length, 0.0)[0]/length, -1.0, atol=1E-3, rtol=1E-3))
A FEniCS Project-based library for simulating thin structures.

4.1 Description

FEniCS-Shells is an open-source library that provides finite element-based numerical methods for solving a wide range of thin structural models (beams, plates and shells) expressed in the Unified Form Language (UFL) of the FEniCS Project.

FEniCS-Shells is compatible with the 2019.1.0 release of the FEniCS Project.

FEniCS-Shells is described fully in the paper:


4.2 Getting started

1. Install FEniCS by following the instructions at http://fenicsproject.org/download. We recommend using Docker to install FEniCS. However, you can use any method you want to install FEniCS.

2. Then, clone this repository using the command:

   ```bash
git clone https://bitbucket.org/unilucompmech/fenics-shells.git
   ```

3. If you do not have an appropriate version of FEniCS already installed, use a Docker container (skip the second line if you have already an appropriate version of FEniCS installed):

   ```bash
cd fenics-shells
./launch-container.sh
   ```

4. You should now have a shell inside a container with FEniCS installed. Try out an example:
```bash
python3 setup.py develop --user
cd demo
./generate_demos.py
cd documented/reissner_mindlin_clamped
python3 demo_reissner-mindlin-clamped.py
```

The resulting fields are written to the directory `output/` which will be shared with the host machine. These files can be opened using Paraview.

5. Check out the demos at [https://fenics-shells.readthedocs.io/](https://fenics-shells.readthedocs.io/).

### 4.3 Documentation

Documentation can be viewed at [http://fenics-shells.readthedocs.org/](http://fenics-shells.readthedocs.org/).

### 4.4 Automated testing

We use Bitbucket Pipelines to perform automated testing. All documented demos include basic sanity checks on the results. Tests are run in the `quay.io/fenicsproject/stable:current` Docker image.

### 4.5 Features

FEniCS-Shells currently includes implementations of the following structural models:

- Kirchhoff-Love plates,
- Reissner-Mindlin plates,
- von-Kármán shallow shells,
- Reissner-Mindlin-von-Kármán shallow shells,
- non-linear and linear Naghdi shells with exact geometry.

Additionally, the following models are under active development:

- linear and non-linear Timoshenko beams,

We are using a variety of finite element numerical techniques including:

- MITC reduction operators,
- discontinuous Galerkin methods,
- reduced integration techniques.

### 4.6 Citing

Please consider citing the FEniCS-Shells paper and code if you find it useful.
along with the appropriate general FEniCS citations.

4.7 Contributing

We are always looking for contributions and help with fenics-shells. If you have ideas, nice applications or code contributions then we would be happy to help you get them included. We ask you to follow the FEniCS Project git workflow.

4.8 Issues and Support

Please use the bugtracker to report any issues.

For support or questions please email jack.hale@uni.lu.

4.9 Authors (alphabetical)

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